Abstract explanations in science*

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'Here it is for the empirical scientists to know the fact and for the mathematical scientists to know the reason why.'
– Aristotle, *Posterior Analytics*, A.13 (79a2-3)

Abstract

This paper focuses on a case that expert practitioners count as an explanation: a mathematical account of Plateau's laws for soap films. I argue that this example falls into a class of explanations that I call abstract explanations. Abstract explanations involve an appeal to a more abstract entity than the state of affairs being explained. I show that the abstract entity need not be causally relevant to the explanandum for its features to be explanatorily relevant. However, it remains unclear how to unify abstract and causal explanations as instances of a single sort of thing. I conclude by examining the implications of the claim that explanations require objective dependence relations. If this claim is accepted, then there are several kinds of objective dependence relations.

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1 Introduction

In the philosophy of science discussions of explanation tend to assume that most explanations are causal explanations. Disagreements arise concerning what a cause is and how the explanatorily relevant causes are to be identified. While it would be foolish to deny that many explanations are causal explanations, in this paper I will question an associated methodological commitment. The standard philosophical methods deployed in discussions of explanation aim to develop an account of causal explanation and then see how widely it can be applied. If recalcitrant cases are found, then either some extended notion of cause is deployed or the case is rejected as not an explanation after all. This methodology more or less guarantees a certain kind of satisfying result: there turns out to be only one kind of explanation. However, the worry remains that this unified category of explanation is more an artifact of the tools used than an underlying unity.

In this paper I work with a different case-driven approach to thinking about explanation. I begin by discussing a case that has been identified as an explanation by expert practitioners. Then I try to figure out what features of this case are responsible for its explanatory import. Finally, I will see to what extent these sorts of cases can be incorporated into some influential theories of explanation.[1] The risk of this approach is that it may turn out that explanations are not all of the same kind. The abstract explanations that I isolate seem to differ in significant ways from the causal explanations that are most often discussed by philosophers of science. If explanations do actually come in two or more kinds, then I think this is just a fact that we as philosophers need to accept. But I will conclude by considering the possibility that a more general conception of explanation can incorporate both causal and abstract explanations. On this approach explanations require objective dependence relations. If we are willing to countenance at least two kinds of dependence relations, then abstract and causal explanations turn out to have more in common than initial appearances suggest.

2 A Case

The case-study for this paper is Plateau's laws for soap-film surfaces and bubbles. Plateau was a nineteenth century scientist who noticed certain regularities in the way soap films formed on wire frames or when soap bubbles came into contact with one another. Here is how Almgren and Taylor summarize the three laws:

> First, a compound soap bubble or a soap film spanning a wire frame consists of flat or smoothly curved surfaces smoothly joined together. Second, the surfaces meet in only two ways: Either exactly three surfaces meet along a smooth curve or six surfaces (together with four curves) meet at a vertex. Third, when surfaces meet along curves or when curves and surfaces meet at points, they do so at equal angles. In particular, when three surfaces meet along a curve, they do so at angles of 120 degrees with respect to one another, and when four curves meet at a point, they do so at angles of close to 109 degrees (Almgren and Taylor [1976], p. 82).

Both sorts of meetings are illustrated in figure 1. Here a wood frame shaped as a regular tetrahedron is used to constrain the soap film. The first sort of meeting is found at any vertex of the tetrahedron. Three surfaces meet at the vertex and the angle between any two adjacent surfaces is 120 degrees. The second kind of meeting is illustrated at the center of the tetrahedron. Here all six surfaces meet at a point. The angle between any two adjacent edges is a bit more than 109 degrees.[2] The power of Plateau's laws is that they work for any such soap-film system. The frame can be any shape. In fact, there might not be a frame at all. An unconstrained soap film may form a bubble that encloses a given volume. Such a system almost trivially meets Plateau's laws as it has only one smooth surface. A more interesting case involves more than one bubble. These systems obey Plateau's laws as well. See figure 2.

[INSERT FIGURE 1]

[INSERT FIGURE 2]

Some remarks about the physics of the situation will indicate what sort of mathematics might help to explain Plateau's laws. A soap film is formed by enclosing a thin layer of water in between two outer layers of soap. The soap molecules orient themselves so that they repel one another across the layer of water. In a stable soap-film configuration, the surfaces necessary to enclose a given volume will have the least area consistent with the given constraints. The area is minimized because this minimizes the surface tension of the system. What is challenging from a mathematical perspective about soap-film cases is that the range of constraints that can be imposed on a system is seemingly unlimited. So it is difficult to survey the surfaces necessary to arrive at the right surface with the least area. This was the problem that Taylor faced in her attempt to mathematically account for the angles mentioned in Plateau's laws. Taylor's solution was to develop a new mathematical theory of surfaces that incorporated these complexities. Her innovations resulted in a novel explanation of this part of Plateau's laws. This is the judgment of mathematician Frank Morgan in his 1996 article "What is a surface?" He points out that the notion of a surface as a "rectifiable current" was unable to solve the problem. Instead, a new conception of the available surfaces was required:

> Physical surfaces such as soap films often consist of pieces of surface meeting along whole singular curves. These curves, although not part of the given boundary, unfortunately count as boundary for rectifiable currents. Explaining the structure of soap films required a new theory of $(\mathbf{M}, \epsilon, \delta)$-*minimal sets* developed by F. Almgren and J. Taylor (Morgan [1996], p. 376).[3]

Prior to Taylor's ground-breaking work in the 1970s, mathematicians were forced to resort to experiments with soap films to discern the structure of this or that surface. This experimental mathematics was publicized in places like Courant and Robbins' *What is Mathematics?* ([1941]). Some philosophers have noted this experimental work as well.[4] There are many complexities here that we are forced to skip. But the essential features of the explanation can be readily summarized.

In their popular article "The Geometry of Soap Films and Soap Bubbles" Almgren and Taylor summarize the explanation in three parts. First, there is a definition of a mathematical analogue of soap films. Each film is treated as a two-dimensional surface embedded in a three-dimensional Euclidean space $\mathcal{R}^3$. Consider a connected network of such surfaces that may or may not involve a fixed frame. It will be *soap-film-like* just in case "it cannot be made to have a smaller area by any small deformation that leaves the frame fixed" (Almgren and Taylor [1976], p. 85). The small deformations are represented by mathematical functions that transform one connected network into another, while leaving the frame fixed. These transformations allow the fusion of two overlapping surfaces into one surface so that all potential changes in area are considered. A *soap-bubble-like* configuration must meet an additional constraint: it encloses one or more volumes so that the net volume is preserved under the relevant small deformations.

With these definitions, the key result is that "any such configuration of surfaces must of mathematical necessity conform exactly to the three geometric principles stated at the beginning" (Almgren and Taylor [1976], p. 86), namely Plateau's three laws. The second step assumes that Plateau's first law holds and shows how the other two follow from it. That is, suppose that a given configuration is soap-bubble-like. Furthermore, suppose that the surfaces are smooth and the total area is finite. Then it follows that the surfaces meet in the two ways allowed by the second and third of Plateau's laws. To get this preliminary result Almgren and Taylor exploit the smoothness of the surfaces to simplify the situation. For any meetings of smooth surfaces, we can cover the meeting region with a small sphere. Within the sphere, the surfaces are flat and the meetings are along straight lines. Almgren and Taylor then argue that this simplified sort of situation requires that the surfaces cross the sphere in a very special way. Each crossing results in a segment of a great circle and the segments must make angles of 120°.[5] The general claim can be illustrated with the soap-film tetrahedron of figure 1. Consider any region where the surfaces meet away from the central vertex. A small sphere centered along the meeting line will have its surface crossed by three half-circles going from one pole of the sphere to the other. See figure 3. A somewhat more complicated configuration of segments of great circles corresponds to the central vertex. Now all six surfaces are involved, so there are six segments of great circles on the surface of the sphere. These segments now meet on three places on the surface of the sphere, but as with the previous case the angles are all 120°. See figure 4. The fact that all the surfaces cross the sphere in this special way follows from the assumption that the configuration is soap-bubble-like. For one can show that a failure to cross in this special way will involve additional surface area that can be removed by a small deformation.

[INSERT FIGURE 3]

[INSERT FIGURE 4]

It may not be immediately clear how many different ways there are for segments of great circles to meet on the surface of a sphere at angles of 120°. It turns out, though, that there are only ten possibilities. So there are at most ten potential ways in which soap-bubble-like configurations can have surfaces meeting. But seven of these ten possibilities can be eliminated by exhibiting small deformations that decrease the overall surface area. For example, consider a potential counter-example to Plateau's laws where a supposed soap film forms on a frame in the shape of a cube and so that all the soap-film surfaces meet at a point at the center of the cube. This would involve twelve soap-film surfaces and correspond to a sphere whose great circle segments met at eight points spaced out over the surface of the sphere. See figure 5. However, one can show that any such configuration is not soap-bubble-like because the surface area can be decreased by a small deformation. The surface area is decreased by adding a small square surface in the middle of the cube. This improved configuration has thirteen surfaces that meet in accordance with Plateau's laws. See figure 6. Other problematic great circle segment possibilities are eliminated by similar means.

[INSERT FIGURE 5]

[INSERT FIGURE 6]

This still leaves the third and most difficult part of the overall proof. This is to show that every soap-bubble-like configuration is composed of smooth surfaces whose total area is finite. Here the resources of the part of mathematics known as geometric measure theory are crucial. In measure theory mathematicians consider functions or "measures" that take the subsets of a set to some output number that reflects the overall size of the subset. In an area minimization problem one needs a "two-dimensional measure" defined over the relevant space, e.g. in our case $\mathcal{R}^3$. In geometric measure theory one uses a surface to directly define a measure over the space covered by that surface. As Almgren and Taylor put it, "a surface that has become a measure assigns to an arbitrary subset of space the amount of the usual area of the surface that is inside that subset" (Almgren and Taylor [1976], p. 92). This might not seem that helpful, but it lets the mathematician get a handle on the measure-theoretic properties of highly counter-intuitive surfaces. So, a survey of all surfaces of this or that sort becomes possible. It is this technique that Taylor exploited in her proof. Proving the existence of soap-bubble-like surfaces was not difficult and these surfaces could then be treated as "solution measures". By using the machinery of geometric measure theory, she was able to show that all such "solution measures" are well-behaved:

> one is able to prove with complete rigor that a solution measure comes from pieces of two-dimensional surface that are smooth on

> their interiors and that the configuration of these surfaces is a legitimate mathematical candidate for a soap-film-like or soap-bubble-like surface. It is paradoxical that knowledge of the manner in which these various surfaces would fit together if they did so smoothly enables one (after considerably more work) to conclude that they do fit together smoothly in that manner (Almgren and Taylor [1976], p. 92).

These mathematical configurations are known as $(\mathbf{M}, \epsilon, \delta)$-minimal sets or almost minimal sets.[6] Taylor's purely mathematical proof established that the almost minimal sets satisfy Plateau's three laws. This explained why Plateau's laws held for actual soap films and soap bubbles due to the intimate connection between these physical systems and the mathematical model.

Although we have skipped many technical details, some key features stand out. There is an initial characterization of the soap-film systems via the physics of surface area minimization. But Taylor's work provided a novel and informative characterization of that very collection. It is novel because it deployed the mathematics of almost minimal sets. And it is informative because the proof shows how the initial description in terms of surface area minimization is linked to the properties of these sets. In particular, the magnitudes of the relevant angles are accounted for by Taylor's calculations. This is why she provided an explanation of these angles that was lacking up to that point. Plateau's experimentation showed that the angles took on these values. But these experiments did not provide any explanation. They did not describe the class of systems in novel and informative terms.

At this stage I do not want to assume that what makes something into an explanation is that it describes a class in novel and informative terms. Different theories of explanation will have different attitudes to this question. As I will make clear in the last section of this paper, I aim to defend an ontic conception of explanation according to which explanations involve objective dependence relations. On this approach, novelty and informativeness are merely evidence that a genuine dependence relation is involved. However, it is a difficult matter to determine when this evidence is conclusive or how generous we should be in populating the world with dependence relations.

A second feature is that the mathematical entities central to the explanation are more abstract than the fact being explained. These are the almost minimal sets. There are many ways to try to capture the idea that one object is more abstract than another. In this paper I will deploy the idea of an instance. A canonical case to keep in mind is the relationship between a type and its tokens. A given type may have many tokens as instances. Exactly what makes a token an instance of this or that type is subject to debate. For our purposes, we need only assume that the instantiation relation in asymmetric. A type has a token as an instance, but not vice versa. Many mathematical structures have concrete systems as instances. The almost minimal sets have soap films as some of their instances, and this is what makes facts about these sets relevant to facts about soap films.[7] While one instance of the abstract surfaces discussed by Taylor are these very soap films, there are many other instances of these surfaces tied to other applications of these mathematical techniques. For example, Almgren and Taylor mention the application of the same techniques to explain skeletons of marine animals known as radiolarians (Almgren and Taylor [1976], p. 92).

So, the appeal to the more abstract objects allows the identification of a property that is necessary and sufficient for the explained property's presence. The purely mathematical surfaces are distinguished from all other mathematical surfaces by being almost minimal sets. These surfaces have the mathematical properties mentioned by Plateau's laws. In particular, when three surfaces meet along a curve, they meet at 120 degree angles and when four curves meet at a point, they meet at (close to) 109 degree angles. The explanation starts by assuming that soap films are instances of the almost minimal sets. Then it shows how this requires the soap films to obey Plateau's laws. So, if a class of physical systems is an instance of these almost minimal sets, then each system in the class obeys Plateau's laws. Furthermore, we have not only a sufficient condition, but a necessary one. If a class of systems obeys Plateau's laws, then the systems are instances of almost minimal sets. For example, suppose a scientist found a class of natural crystals that were all found to obey Plateau's laws. It would be the case that they were also instances of almost minimal sets. Being an instance of an almost minimal set is both necessary and sufficient for satisfying Plateau's laws.

It might seem like the claims of the last paragraph are vulnerable to the following objection. Suppose that a teacher builds a plastic model to illustrate Plateau's laws. This involves shaping the plastic so that the configuration of the surfaces and their angles of contact matched Plateau's laws. However, Taylor's argument would not explain why this plastic model fit Plateau's laws. I agree with this, but it is important to see that it does not undermine my account of how abstract explanations explain. In the soap-film case we begin with an initial description of the class of surfaces that makes reference to area minimization. This is why it is explanatory to bring to bear Taylor's theory of almost minimal sets. These minimal sets make the right kind of link to facts about area minimization. That is, they indicate that there is an objective dependence relation between facts about the sets and facts about soap-films. By contrast, the plastic model is not part of a collection of things that naturally take on minimal surface configurations. So we cannot explain its arrangement by appeal to the almost minimal sets. Instead, we explain its arrangement by appeal to the actions and goals of the teacher who built it. This shows that merely having a necessary and sufficient condition for the explained property is not enough for explanation. To have an explanation the explained property must also be appropriately linked to the condition. This link obtains in the soap-film case, but not in the case of the plastic models. For a naturally occurring crystal, it would not be immediately obvious whether or not the Taylor explanation applied. What would have to be settled is whether or not the formation of the crystals fit some area minimization principle. If it did, then Taylor's explanation could be extended to the crystals. If it did not, then the fit with Plateau's laws would require a different sort of explanation or perhaps be unexplainable.

3 Abstract and Causal Explanations

I have identified three key features in the Taylor case. We think we have an explanation when we have found a (i) classification of systems using (ii) a more abstract entity that is (iii) appropriately linked to the phenomenon being explained. Whenever an explanation has these three features I will say that we have an abstract explanation. In this section I want to argue that the best, recent work on causal explanation is not able to naturally accommodate these abstract explanations. Still, I will highlight a sort of non-causal dependence that is operating in these cases that generalizes causal dependence. An easy way to see that causal dependence is not involved would be by emphasizing the highly mathematical character of the Taylor case. As nobody thinks that causal relations obtain in pure mathematics, we have a non-causal explanation. But this does not help us to see what is valuable in an abstract explanation or what non-causal dependencies might be involved. That is why I will discuss two goals for causal explanation that have been emphasized by Woodward and Strevens, respectively. The first feature is what I will call "fixed-object" information, while the second is what Strevens calls cohesion. Abstract explanations lack both of these goals, and this is central to their value to science.

Woodward is at pains to allow for explanations via causal generalizations that are not laws. These generalizations may lack the necessity or universal scope that many require laws to have, but yet they can still explain. For Woodward (writing with Hitchcock) these generalizations explain because they say how an object would change under this or that intervention: "a generalization must describe a relationship that holds for certain *hypothetical* values of X and Y *possessed by the very object* o, namely those values of X and Y possessed by o in the hypothetical situations where the value of X is changed by an intervention" (Woodward and Hitchcock [2003], p. 20). For example, we may have a generalization that relates plant height to available water and sunlight. This generalization applies to a given plant if it fits not only the actual values, but some range of other values for water and sunlight. The generalization must state how these different values would have led to different heights for that very plant. For this reason, I will say that Woodward-style causal generalizations provide *fixed-object* information. We hold fixed the object mentioned in the explanandum and must be told how it would be different in appropriate counterfactual scenarios.

There are of course many situations where fixed-object information can be scientifically valuable, and so it is reasonable for Woodward to focus on generalizations that provide this sort of information. However, the Taylor case is not easily approached in these terms. If this case does involve an explanation, then there are explanations where fixed-object information is not mandatory. The value of the Taylor case lies instead in the *other-object* information that it provides. With Taylor's proof we get a necessary and sufficient condition for a soap film to hold. It tells us, then, in what respects the class of soap films are similar to one another and also different from all those things that are not soap films. This does not tell us how any particular physical system will change under an intervention. So, there is no fixed-object information in the explanation. Instead, we learn that all these systems are instances of almost minimal sets. They are similar in this respect, and different in that very way from all those physical systems that fail to obey Plateau's laws.

Woodward could respond by granting the scientific value of this sort of information, but also by insisting that this case is not an explanation. Here it is important to remember the case-driven methodology that I am deploying. We begin with a case that practitioners say is an explanation and try to figure out what features of the case are prompting them to do that. In fact, Woodward seems open to non-causal explanations where his notion of intervention does not apply. All Woodward requires is something like what our case provides: an answer to a question about similarities and differences, even when those features are not tied to a fixed object. He considers a case where the stability of planetary orbits can be explained by appeal to the dimensions of space-time. But it does not make sense to interpret this explanation as telling us about fixed-object interventions. He also mentions, but does not endorse, Steiner's discussion of the difference between explanatory and non-explanatory proofs in mathematics. Summing up, Woodward says

> When a theory or derivation answers a what-if-things-had-been-different question but we cannot interpret this as an answer to a question about what would happen under an intervention, we may have a noncausal explanation of some sort (Woodward [2003], p. 221).

In our case, the question that is answered provides other-object information. There is no discussion of how a given soap film would change if it was not an instance of an almost minimal set. All we learn is the systematic relationship between the more abstract objects and their properties and the more concrete objects and their properties. It appears that Woodward could allow these sorts of explanations, then, if the sort of explanatory question here could be suitably clarified.[8]

Suppose we have assumed that explanations turn on dependence relations. An attractive aspect of Woodward's approach to causal explanation is that he can trace the value of explanations to the description of causal dependence relations. We can generalize Woodward's approach by taking the essential features of abstract explanation to correspond to an *abstract* dependence relation. That is, we can say that a soap-film surface obeying Plateau's laws depends on its being an instance of an almost minimal set. As with causal dependence, there can be subtle questions about exactly how we should interpret talk of abstract dependence. For example, are such facts completely objective, or is there a context-sensitivity to facts of abstract dependence? I return to these issues briefly in section 5. For now it will not do any harm to talk as if abstract dependence relations obtained in the world, and that they are described by abstract explanations.

Strevens allows important explanatory contributions from more abstract entities like mathematical objects. But he ultimately requires these entities to represent underlying causal processes: "The ability of mathematics to represent relations of causal dependence – wherever it comes from – is what qualifies it as an explanatory tool" (Strevens [2008], p. 331). Even an explanation of a pattern like our explanation of Plateau's laws must meet a demanding condition that Strevens labels cohesion (Strevens [2008], p. 110). Cohesion relates roughly to the degree to which the ultimate causal details of the events covered by the more abstract claims are physically similar. The clearest account of it that Strevens gives is his notion of dynamic contiguity. This is presented as a necessary condition on cohesion: "every causal process . . . corresponds to a trajectory in the state space of fundamental physics . . . Take the trajectories corresponding to every one of a model's concrete realizers, each a thread in state space. If this set of state space threads is contiguous, the model is *dynamically contiguous*" (Strevens [2008], p. 105). Dynamic contiguity blocks the explanation of a pattern that is due to causally heterogeneous processes. From a causal perspective, this is worse than mere multiple realizability. Certain sorts of multiply realizable processes will still be dynamically contiguous because between any two systems realizing the process, there is a continuous chain of other such systems in the state space of fundamental physics. But if there are gaps, then the causal processes are too different to be encompassed by a single causal model, no matter how abstract.

I claim that the explanation discussed earlier violates Strevens' cohesion requirement. Recall that Plateau's laws apply to soap films of any sort and also to frames composed of any material and widely varying in structure, thickness and material composition. So, there are two ways that this class of systems fails to be dynamically contiguous and so fails to be cohesive. First, the molecular details of soap itself and the water-soap solution making up the soap films will block contiguity. Soap molecules must have a long hydrophobic (water repelling) end and a short hydrophilic (water attracting) part at the other end. This is critical to the role of the soap molecules in constraining a thin layer of water. But beyond these functionally specified requirements, there are few requirements on what a soap film can be made of. Synthetic detergents can also be used to make soap films despite their chemical differences from traditional soap. It is not likely that these materials will lead to state space trajectories that are contiguous at the level of fundamental physics. For example, the trajectories of soap-film processes made of Dawn liquid dish soap are sure to be isolated from the trajectories of soap-film processes composed of competing brands, not to mention traditional soap. Second, and perhaps more importantly, the composition and structures of the frames are too heterogeneous for contiguity to obtain. Some frames will be made of wire, while others will be made of wood. Some frames will be tetrahedrons, while others will be twisted knot-like structures. So, Taylor's explanation of Plateau's laws is not going to satisfy Strevens' cohesion requirement. Despite being an explanation of a physical pattern, the explanation is not a causal explanation in Strevens' sense.

As with Woodward, Strevens could reject our cases as cases of genuine explanations. Sometimes Strevens talks as if our intuitive judgments are decisive (Strevens [2008], p. 38). But this violates our case-driven methodology. It is expert practitioners who should guide our judgments on cases and influence our philosophical theory of explanation. At other places Strevens expresses the hope that his account of explanatory relevance could be extended quite broadly to include "mathematical, moral, and aesthetic explanation (and possibly also noncausal explanation in science, if there is such a thing)" (Strevens [2008], p. 5). He does provide an extended discussion of what many call equilibrium explanations, and seems to link the shape of a single soap bubble to this sort of case (Strevens [2008], pp. 266, 272, 331). Strevens must think that all the cases he discusses satisfy his cohesion condition as he shows no hesitation when arguing that his theory of causal explanation fits these cases. But based on the above argument, I conclude that the explanation of Plateau's laws requires giving up Strevens' cohesion requirement. Sometimes, it seems, it is important for a scientific explanation not to be cohesive. This is central to the explanation's ability to carve up systems into two kinds via the properties of the more abstract objects. Cohesion is just not mandatory or even desirable if this is what the explanation aims to accomplish.

So here we see another feature that is distinctive of abstract dependence relations. They obtain between kinds of things even when processes involving those kinds of things fail to be cohesive. This can seem mysterious if we assume that dependence relations all bottom out in fundamental causal facts. But there is little reason to make this assumption at the beginning of an investigation of explanation, and it appears that explanatory practice contradicts this assumption. The upshot of our discussion is that central features of causal explanations must be dropped to make sense of our abstract explanations. These abstract explanations provide other-object, not fixed-object, information. They violate a cohesion requirement and thereby obtain a certain sort of classificatory power that would be missed if one were forced to group systems causally. Both virtues of these abstract explanations, then, move them away from the virtues of causal explanation identified by Woodward and Strevens. This strongly suggests that we have two different kinds of explanation, with different aims and benefits for science.[9]

4 Recent Work on Mathematical Explanation

There is a growing literature on the benefits that mathematics provides for scientific explanation. These discussions have often been motivated by explanatory indispensability arguments for platonism. However, independent of debates about those arguments, it is important that we see to what extent mathematics might be helping with explanations even when it is not tracking causes in either of the ways that Woodward or Strevens would require.[10] Perhaps the most natural suggestion along these lines is Lyon's recent claim that some mathematical explanations are program explanations in the sense of Jackson and Pettit. Jackson and Pettit distinguish between a causally efficacious property and a causally relevant property. A causally efficacious property is one whose bearer thereby gains certain causal powers, e.g. to bring about a certain effect in a given situation. Clearly, one sort of causal explanation would explain that effect by noting the presence of the appropriate causally efficacious property. However, Jackson and Pettit insist that not all causally relevant properties are causally efficacious. Here they emphasize "abstract properties" and insist that "[a]lthough not efficacious itself, the abstract property was such that its realization ensured that there was an efficacious property in the offing ... The realization of the higher-order property ... meant that there would be a suitably efficacious property available" (Jackson and Pettit [1990], p. 114). When the appropriate ensuring or guaranteeing relation obtains, we can cite the abstract property in an explanation of the properties that it indirectly causes.

Lyon argues that mathematical explanations of physical phenomena are program explanations. He notes one instance of the application of Plateau's laws: in a rectangular frame, the soap film forms with four vertices where four curves meet at the requisite angle of close to 109 degrees. See figure 7. Lyon notes that "it is a mathematical fact that the surface bounded by the frame that minimizes surface area is the surface depicted" (Lyon [2012], p.

564). Lyon goes on to describe the explanatory benefits of this sort of appeal to abstract properties. Citing the particular causally efficacious property in play in this or that soap-film system will not indicate that the configuration was more or less inevitable. But the program explanation reflects this modal fact:

> pretty much no matter what the precise sequence of movements was, Plateau's soap film would still have formed . . . Changing the exact locations and movements of the soap molecules would not change their final macroscopic form. The mathematical explanation involving the mathematical fact from the theory of minimal surfaces gives us this modal information (Lyon [2012], p. 567).

A program explanation does better than an ordinary causal explanation because it provides modal information about the phenomenon that would otherwise be lacking. In fact, this configuration of soap films was more or less inevitable, and we want to include this fact in our explanation and not just describe it as if it was highly contingent.

[INSERT FIGURE 7]

Program explanations retain a tie to causal explanations via the required link between the abstract, programming property and the concrete, causally efficacious properties. However, program explanations are much easier to achieve than the abstract explanations I have identified. Recall that these abstract explanations explain in a special way. Program explanations and abstract explanations both appeal to what is more abstract than the phenomenon being explained. However, abstract explanations invoke a more abstract entity and its properties. Program explanations appeal only to a more abstract property of the physical system itself. This might not seem like such a big difference, but it has important implications for the features that are central to the explanatory value of abstract explanations. We get necessary and sufficient conditions for the explained property to apply as well as an informative comparison between novel kinds of objects. A programming explanation need only cite a sufficient condition for the explained property to apply. The sufficiency here is some sort of mathematical or logical guarantee that the explained property would apply. As such it may include redundant conditions and still qualify as part of a program explanation. Furthermore, we might have sufficient conditions for the property to apply in that very case without arriving at any informative comparison between cases where that property applies and cases where it fails to apply.[11]

So, while it appears that all abstract explanations will accomplish the aims of a program explanation, some program explanations will not contain the features that make abstract explanations scientifically valuable. This comes through fairly clearly in Lyon's own brief discussion of Plateau's laws. To provide a program explanation of the configuration in figure 7 it is enough to note that the system minimizes its surface area. This does in fact guarantee that the system will have the configuration it does. But the explanandum in our case was the phenomenon of the wide applicability of Plateau's laws. This cannot be explained by noting only that each system minimizes its surface area. It is only with the appeal to the mathematical theory of almost minimal sets that we have explained the angles. Taylor's explanation provides a novel classification of these soap-film systems and thereby explains the application of Plateau's laws. We learn what the systems have in common and how this commonality is responsible for the holding of Plateau's laws. We also come to know what other systems that fail to obey Plateau's laws are like and how they differ from the soap-film systems. The exhaustive classification is only possible because the abstract explanation isolates an informative necessary and sufficient condition for this property to apply. As I put things earlier, we learn about abstract dependence relations. This generates an explanation of each particular case that has the modal information that Lyon values. But it provides additional information that Lyon does not mention. Of course, it is possible for Lyon to deny that this sort of information is important to explanation. However, the case-driven methodology I am using suggests that these features are important, at least in some cases. Other cases would need to be considered to see to what extent they fit with mere program explanations or the more demanding abstract explanations.[12]

Lange has also developed an account of mathematical explanation in science. He isolates a class of "distinctively mathematical explanations" in science. Lange is clear that the value of these explanations is that they do something that causal explanations cannot do: "these [distinctively mathematical] explanations work not by describing the world's network of causal relations in particular, but rather by describing the framework inhabited by any possible causal relation" (Lange [2013], p. 509). An example of such a framework would be Newton's second law $F = ma$. This does not describe the operation of any particular cause or force, but is instead "the framework within which any force must act" (Lange [2013], p. 503). As one might put it, for something to be a force, it must satisfy Newton's second law. Other framework principles will be tied to what is contextually salient in a given explanatory context. For example, Lange discusses the explanation for why a person cannot carry out a certain crossing over the bridges of Königsberg. The purely mathematical part of the explanation pertains to abstract topological structure (Lange [2013], p. 489). But the rest involves "various contingent facts presupposed by the why question that the explanandum answers, such as that the arrangement of bridges and islands is fixed" (Lange [2013], p. 506).[13] There is thus a different grade of modality for these mathematical explanations. Supposing that causal relations are merely physically necessary, mathematical explanations show that, given that certain things are held fixed, the phenomenon is more than physically necessary. It is necessary to whatever degree mathematical truths are necessary.

Lange's distinctively mathematical explanation relation is thus much stronger than Lyon's programming relation. The programming relation between an abstract property and its causally efficacious realization is quite strong. However, the operation of the causally efficacious property is just physical necessity. So Lyon's explanations will at most show that some phenomenon is physically necessary. By contrast, Lange is in position to explain phenomena in a way that shows them to be mathematically necessary (at least under a certain description). This makes Lange's account more demanding than Lyon's and so Lange is not vulnerable to as many counterexamples. However, there still seem to be two barriers to using Lange's approach to make sense of our abstract explanations.

The first problem with Lange's proposal is that he allows the explanatory question to hold fixed certain facts, even when those facts are highly contingent. This is reminiscent of van Fraassen's pragmatic approach to explanation, where explanations are answers to why-questions and why-questions incorporate a range of pragmatically motivated factors. Lange must provide some further story of how the context shapes or constrains genuine explanatory questions. Otherwise, the danger is that every phenomenon will have a distinctively mathematical explanation and the special sort of case that Lange is after will be obscured. A hint of the problem is apparent in the two cases just mentioned. An explanation that assumes Newton's law of motion is assuming something about the essence of force. But an explanation that assumes something about the arrangement of the bridges of Königsberg is not making an assumption about some fixed essence or nature. Of course, there might be a way to restrict what is available to be held fixed in this sort of explanation that would avoid the trivialization problem. Without such a restriction it is not clear how the modal value of the explanation is to be cashed out.

The second problem is that Lange is working with the idea that an explanation need only cite some sufficient conditions for the phenomenon being explained. When the conditions are not also necessary, there is a risk that redundant conditions will be included. These conditions will not undermine the modal strength of the entailment, so it is not clear why Lange would say they undermine the goodness of the explanation. But we have seen what explanations with merely sufficient conditions are unable to do. They are not able to provide an informative characterization of the systems in question that highlights their similarities amongst themselves and their differences with other systems. In the Königsberg bridges case, the feature that provides a necessary and sufficient condition for the right sort of tour of the bridges is that there is an *even* number of connections to any node in the network. So a sufficient condition on there failing to be this sort of tour is that one node has five connections. Consider a system like the bridges of Königsberg with a node with five connections. An explanation that appeals to some node not being even and another explanation that appeals to the one node with five connections are equally modally strong. So they should be equally good explanations for Lange. But it is only the first that qualifies as an abstract explanation in our sense.

In summary, both Lyon and Lange understate the importance of a novel and informative classification of physical systems. I have so far claimed that these classifications are central to the value of abstract explanations and also that abstract explanations describe a special kind of abstract dependence. If there are causal and abstract dependence relations in the world, then we want scientific explanations that reflect both.

5 Explanation and Dependence

I have presented a case that practitioners describe as an explanation and argued that the most prominent extant approaches to causal and mathematical explanation are unable to accommodate these cases. The features that I have emphasized distinguish them from more ordinary causal explanations. One weak conclusion, then, is that there are different kinds of explanation. Or, as I have put it so far, there are two kinds of dependence relations. In this section I will consider in more detail the claim that explanations require objective dependence relations. If this claim is accepted, then we can have as many kinds of explanation as there are dependence relations. Some might be reluctant to allow the abstract dependencies described by abstract explanations, but it is not clear what a viable alternative would be.

According to Salmon's classic discussion, conceptions of explanation can be divided into epistemic, modal and ontic approaches (Salmon [1984], ch. 4). These conceptions disagree on what explanations are aiming to do. An epistemic approach links explanations with states of knowledge. There are many ways to do this, but the most influential version of the epistemic approach is Hempel's approach. On this view, an explanation of E must be an argument that would have correctly predicted E in advance. Few defend an epistemic conception today, primarily because the link between explanation and prediction seems too confining. Especially in indeterministic contexts, a good explanation may not be tied to potential prediction.

Salmon himself advocates an ontic conception that ties explanations to features of the world: "The ontic conception sees explanations as exhibitions of the ways in which what is to be explained fits into natural patterns or regularities" (Salmon [1998], p. 320). The more specific ontic approach that Salmon defends is of course a causal approach. Woodward and Strevens are thus properly seen as defenders of an ontic approach as well. By contrast, Lange argues against the ontic conception using his mathematical explanations: "I have argued that the modal conception, properly elaborated, applies at least to distinctively mathematical explanation in science, whereas the ontic conception does not" (Lange [2013], pp. 509–510). On a modal approach "scientific explanations do their jobs by showing that what did happen had to happen" (Salmon [1998], p. 320). There are objective modal facts that certain explanations uncover and account for.

I suggest that a modified ontic approach to explanation can help one to see what things as different as our causal and abstract explanations have in common. In the causal case, accounts of causal explanation like Woodward's view clarify how some outcome causally depends on its explanatorily relevant causes. For example, a cause of the height of some plants is the amount of water the plants received. A causal explanation of the height of these plants could involve a generalization that included a variable for the water received. This explanation requires that there be an objective dependence relation between the water received and the height of the plants. Woodward uses his account of manipulations and interventions to make precise what these dependence relations come to. My proposal is that an abstract explanation requires something similar. In addition to causal dependence relations, there are what we could call abstract dependence relations. In the Taylor case, the satisfaction of Plateau's laws depends abstractly on the features of the almost minimal sets. As with causal dependence, abstract dependence is said to be completely objective. These sorts of relations obtain in the world, and it is the job of science to discover and theorize about them. In both the abstract and the causal case, an explanation requires more than merely describing some phenomenon. What is added to the description to obtain the explanation is the right kind of dependence relation.

At least two things are needed to flesh out this suggestion. First, there should be a theory of what abstract dependence comes to and how these relations are distributed in the world. Without this theory there is a serious problem with overgenerating abstract explanations. Unless there is some principled way to constrain the proliferation of abstract dependence relations, there will be too many of them and so the value of abstract explanation will be diluted. Developing such a theory is no easy task. The availability of slight variants on our almost minimal sets is one example of the problems that this theory would face. It is possible to define distinct, though similar, kinds of mathematical entities and relate these "pseudo-minimal sets" to soap films in ways that mimic Taylor's proof. But the mere existence of these alternatives should not lead us to conclude that new abstract dependence relations obtain here as well. More investigation is needed, then, to see how to block this overgeneration worry.

A second component for this modified ontic approach is an account of how we can know that abstract dependence relations obtain. As with the causal case, there are significant problems in saying how we can know that an objective dependence relation holds. If we fail to know about these relations, then we cannot know the explanations. The most popular strategy for overcoming this sort of skepticism is some kind of inference to the best explanation. As suggested earlier, the informativeness of a novel classification might itself be the evidence that a dependence relation obtains. But so far this is just a suggestion. It is not likely that we will find a worked-out epistemology in the scientific practice that guides our investigations. A genuine theory of abstract explanation would need some new epistemological components as well.

Philosophers of science accustomed to focusing on causal dependence relations might balk at invoking any new dependence relations. Such a philosopher could accept that

C explains E iff E depends on C

but insist that all dependence is causal dependence. Against this sparse picture I would point to a third sort of dependence relation that has been the subject of intense scrutiny in recent work on metaphysics. Consider, for example, how Koslicki concludes her recent paper "Varieties of Ontological Dependence":

> an explanation, when successful, captures or represents (...) an underlying real-world relation of dependence of some sort which obtains among the phenomena cited in the explanation in question ... If this connection between explanation and dependence generalizes, then we would expect relations of ontological dependence to give rise to explanations within the realm of ontology, in the sense that a successful ontological explanation captures or gives expression to an underlying real-world relation of ontological dependence of some sort (Koslicki [2012], pp. 212-213).[14]

One variety of ontological dependence that Koslicki isolates is what she calls constituent dependence (Koslicki [2012], p. 205). For example, for lightning to occur is just for energy to be discharged by some electrons in a certain way, and when lightning occurs, these electrons are constituents of the lightning. Constituent dependence is tied to a special sort of explanation where we explain by appeal to what makes up or constitutes what is being explained. Koslicki extends constituent dependence beyond simple mereological cases. For example, she suggests that sets stand in a constituent dependence relation to their members.[15] I do not wish to take a stand on the viability of this specific account of ontological explanation. All I aim to do is to indicate the general link between explanations of a given sort and some kind of objective dependence relation. This should help to make palatable the conditional claim that if we have a special sort of abstract explanation, then there is also a special sort of abstract dependence relation. Restricting all dependence to causal dependence is implausible.

There are compelling reasons to distinguish between constituent and causal dependence. The causes of an event need not be its constituents, and the constituents of an event need not be its causes. In section 3 I have argued that abstract dependence is distinct from causal dependence.[16] But it might be tempting to try to reduce abstract dependence to constituent dependence. It is hard to argue that this cannot be done, but there is one prima facie barrier that seems difficult to overcome. Note that a distinguishing feature of abstract explanation and abstract dependence is that we appeal to a more abstract entity that has a more concrete entity as an instance. In the Taylor case, the almost minimal sets have some soap-film systems as instances. By contrast, in the constituent cases, the entities that we appeal to in the explanation are constituents of the fact to be explained. The constituents of the lightning help to explain the presence and features of the lightning. In these mathematical cases, it seems clear that a more abstract entity is not a constituent of a more concrete entity. It follows that whatever relation obtains in the abstract case is distinct from the relation that obtains in the constituent case.

6 Conclusion

In this paper I have deployed a case-based approach to explanation. After identifying a case from the practice of science, I argued that there are three features that seemed to render it both different from causal explanations and valuable. This suspicion was reinforced through our discussion of Woodward and Strevens. Our case fails Woodward's fixed-object condition and Streven's cohesion requirement. This suggested a notion of abstract dependence alongside causal dependence. I then turned to a critical discussion of recent work on mathematical explanation by Lyon and Lange. Neither approach seems able to discern the special value of abstract explanation. But it is possible to subsume both abstract and causal explanations under a version of the ontic approach to explanation. There is clearly still much to be learned about explanation by reaching back into the practice of science and mathematics. A case-based approach promises to help us arrive at a more comprehensive understanding of the variety and potential underlying unity of explanation.

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Notes

[1]I do not claim that this approach is particularly original. In fact, it seems to represent a new consensus for work on explanation, especially for cases where mathematics seems to be playing an important explanatory role in science. See, in particular, (Batterman [2002], [2010]) and (Rice [forthcoming]).

[2]The exact value is $\cos^{-1}(-\frac{1}{3}) \approx 109.47^\circ$.

[3]See also an earlier paper where the relevant section is called "The Mathematical Explanation" (Morgan [1990], p. 99).

[4]See (Nagel [1961], p. 392, fn. 24), (Sober [2011]), (Lyon [2012]) and (Saatsi [2012]). As far as I know, no philosophers have investigated the significance of Taylor's work.

[5]A great circle of a sphere results from a plane that goes through the center of the sphere.

[6]The most accessible discussion of the key ideas of Taylor's proof in (Taylor [1976]) that I have been able to find is (David [2013]). David defines an almost minimal set in more or less the way that Morgan defines his $(\mathbf{M}, \epsilon, \delta)$-minimal sets in (Morgan [2008]). I will use the former term from now on. In

David's presentation of Taylor's proof the crucial step is to define a notion of density of a region of a two-dimensional surface in $\mathcal{R}^3$. One can determine the density of a region that looks like a plane or one of the two sorts of meetings permitted by Plateau's laws. The proof then shows that every region of an almost minimal set looks like one of these types by calculating its density. More precisely, each region is C^1-diffeomorphic to one of these types. See (David [2013], §§3, 4).

[7]Some prefer to treat the instantiation relation as basic, but I believe a case can be made that many examples can be analyzed further using some notion of structure-preserving mapping. Unfortunately, I cannot pursue these subtleties here, but will reserve these questions for future work.

[8]See (Rice [forthcoming]) for one suggestion about how to extend Woodward's account somewhat along these lines.

[9]This does not show that all abstract explanations are non-causal explanations. Sometimes we will have a classification using a more abstract entity that is appropriately linked to the phenomenon being explained and either Woodward's or Strevens' conditions on explanation will also be met. My point in this section is that neither Woodward's nor Strevens' conditions are met in the Taylor case and this highlights what is special about these abstract explanations.

[10]See, e.g., (Colyvan [2012], esp. ch. 5), for a recent survey. (Batterman [2010]) pursues these issues independently of their tie to indispensability arguments.

[11]Strevens ([2008], p. 249) provides similar criticisms of program explanation and uses the problems to motivate his own causal requirements.

[12]It is of independent interest to determine to what extent these competing notions of scientific explanation can be extended to handle explanations in pure mathematics. See, for example, (Mancosu [2008]) and (Lange [2010]). Our case seems to provide support for Baker's recent criticism of Steiner: a mathematical scientific explanation need not involve an explanatory proof of a purely mathematical claim (Baker [2012]). I hope to pursue these issues in future work.

[13]See also (Lange [2013], p. 497).

[14] See also (Correia [2008]), (Rosen [2010]) and (Bennett [2011]) for further discussion and references.

[15] See also (Caplan, Tillman and Reeder [2010]).

[16] After this paper was largely completed I came across (Skow [forthcoming]). Skow argues that "all explanations of events other than in-virtue-of explanations are causal" ([forthcoming], p. 4), where in-virtue-of explanations are something like constituent explanations. Skow develops a theory of causal explanation that is different from both Woodward's theory and Strevens' theory. In particular, "ruling out causal histories is sufficient for being causal-explanatory" ([forthcoming], p. 14). I must reserve engagement with this important paper for future work.

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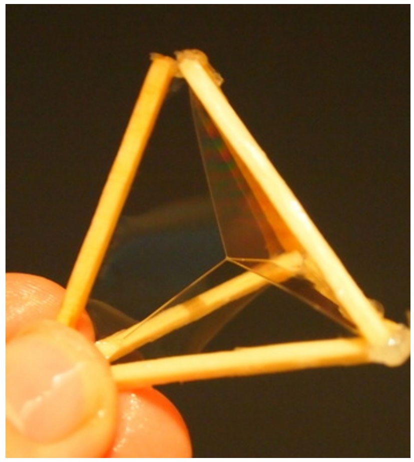

Figure 1: Soap film on tetrahedron

Figure 2: Three bubble cluster

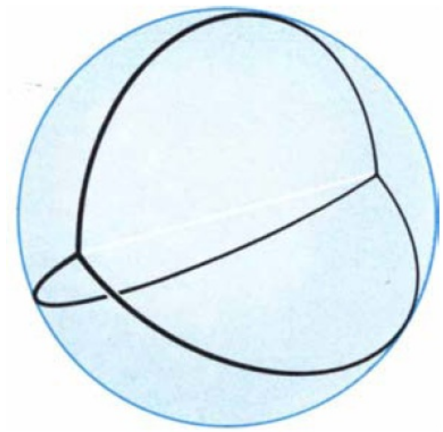

Figure 3: Segments of great circles (Almgren and Taylor [1976], pp. 90–91)

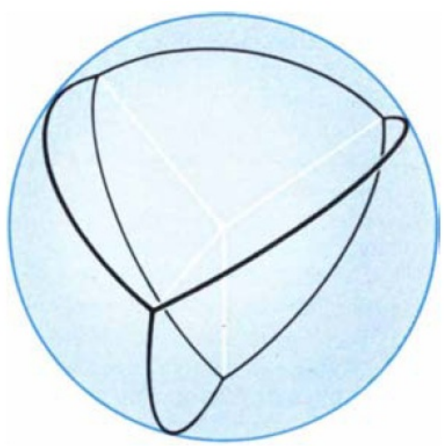

Figure 4: Segments of great circles for tetrahedron vertex (Almgren and Taylor [1976], pp. 90–91)

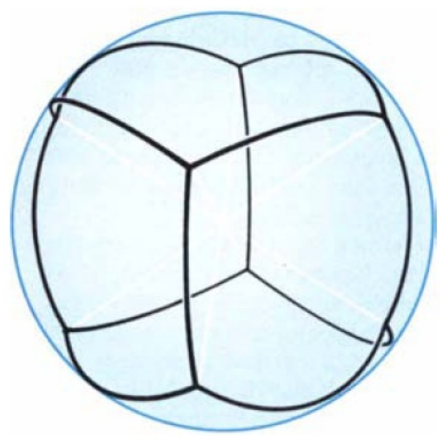

Figure 5: Segments of great circles for twelve surfaces (Almgren and Taylor [1976], pp. 90–91)

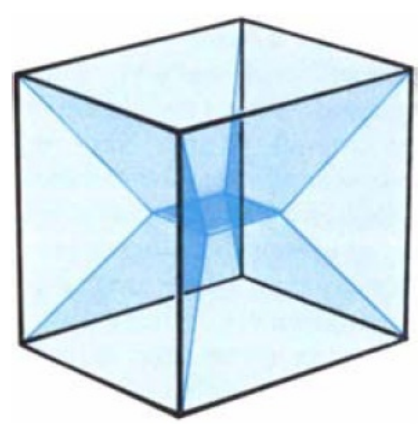

Figure 6: Deformation to improve on twelve surface configuration (Almgren and Taylor [1976], pp. 90–91)

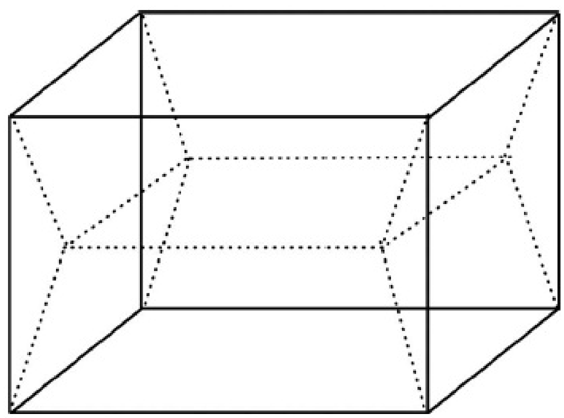

Figure 7: An instance of Plateau's laws (Lyon [2012], p. 563)

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