Review of[1]
Proof and Other Dilemmas: Mathematics and Philosophy
edited by Bonnie Gold & Roger A. Simons
Spectrum Series, MAA, 2008
346 pages, Hardcover

Review by
Christopher Pincock (pincock@purdue.edu)
Department of Philosophy, Purdue University, West Lafayette, IN 47907

1 Introduction

This book is made up of 16 newly commissioned essays by prominent philosophers of mathematics along with some mathematicians and mathematics educators who have reflected on philosophical aspects of mathematics. As Bonnie Gold, one of the editors, explains in her introduction, the point of the volume is "to increase the level of interest among mathematicians in the philosophy of mathematics" (p. xiii). This goal seems to have led the editors to shy away from contributions that focus on the usual foundational debates such as the the debates between logicism, formalism and intuitionism from the first half of the twentieth-century. Also missing is much discussion of the ongoing work on logic and set theory that is often deployed in philosophical discussions of the nature of mathematics. Instead the editors focused on commissioning articles that would be accessible to mathematicians and advanced students of mathematics and that would also hopefully engage their interests. Most contributors responded by summarizing a central problem in the philosophy of mathematics, typically with an emphasis on their preferred take on the issue. The result is a very useful attempt to open more of a dialogue between philosophers and mathematicians.

2 Summary

The first part of the book is "Proof and How it is Changing". It contains articles by the philosopher Michael Detlefsen and the mathematicians Jonathan Borwein and Joseph Auslander. Auslander takes up the traditional position that deductive proof is a central feature of mathematics, but also admits that what counts as a proof may change over time as the mathematical community responds to mathematical and non-mathematical developments. He is also keen to emphasize that proofs have many functions beyond simply justifying a belief in a given theorem. These additional functions include providing an explanation of a theorem and helping to explore a new domain. Auslander goes on to criticize Zeilberger's call for a new "semi-rigorous mathematical culture" as "quite wrongheaded" (p. 71). This sets up an important contrast with Borwein's discussion which champions an "experimental mathematics" where computer-assisted exploration of mathematical results takes a central place. Borwein gives a list of eight roles for computers in mathematics which include not only "confirming analytically derived results", but also "exploring a possible result to see if it merits formal proof" and "discovering new facts, patterns and relationships" (pp. 44-45). These different roles are then illustrated with several intriguing examples.

Detlefsen takes up the place of proof in mathematics in his contribution, but in a way that reflects the relative priorities of philosophers as compared to mathematicians. This is reflected in Detlefsen's focus on clarifying the claim that computers are changing the nature of proof. He does this mainly by engaging with the arguments of other philosophers. He concludes that these arguments have not been successful and so it is premature to see some sort of revolutionary shift in our conception of mathematics. Similarly cautious results are obtained from Detlefsen's discussion of the role of diagrams in proof. While he concedes that some "insightful cases have been made for the significance of diagrammatic reasoning as justificative", it remains the case that "our understanding of possible limits on justificative uses of diagrammatic reasoning have been similarly advanced" (p. 27). The contrast here between the philosopher and the mathematicians runs throughout the volume. Most philosophers place a premium on clarifying a question and considering the pros and cons of the arguments that answer it. Mathematicians, at least as reflected in this volume, seem more eager to marshal examples that they think clearly favor their respective positions.

This dichotomy is especially prominent in the second part of the book focusing on social constructivist approaches to mathematics. The philosopher Julian Cole spends most of his article motivating a specific version of a social constructivist interpretation of mathematics according to which human actions bring abstract mathematical entities into existence. This position is hardly ever discussed in contemporary philosophy of mathematics despite its apparent popularity among some mathematicians. Cole complains, though, that Ernest's book defending social constructivism seems to have "no argument" and that the two arguments he can find in Hersh's discussion are not convincing (pp. 121-122). This leads Cole to offer an argument that his version of social constructivism is superior to its main competitor, namely platonism. Platonism posits abstract objects which exist independently of human actions, but Cole complains that "Platonistically construed mathematical domains are explanatorily and justificationally superfluous. Consequently, we should not accept their existence" (p. 125).

The mathematicians Philip Davis and Reuben Hersh provide articles that also support their respective versions of social constructivism. Davis's contribution considers the question "When Is a Problem Solved?" His answer is basically that a problem is never really solved: "meaning is dynamic and ongoing and there is no finality in the creation, formulation and solution of problems, despite our constant efforts to create order in the world" (p. 83). However, this somewhat pessimistic opening position is further refined as Davis considers several different kinds of mathematical problems and how a mathematician may decide that they have been more or less solved, at least for present purposes. Hersh's essay begins with the bold assertion that "Mathematical entities do exist, they are cultural items" (p. 96). This is followed by an investigation of how we might distinguish mathematics from other domains. Hersh concludes that the objects of mathematics are "all those abstractions that lend themselves to conclusive, irresistible reasoning" (p. 100). He then argues that the empirical study of humans using "history, sociology, anthropology, psychology, cognitive and neuroscience" (p. 101) is necessary to better appreciate what these abstractions are and how such conclusive reasoning is possible.

Philosophers dominate part three of the volume focused on the nature of mathematical objects. A review of the four essays by Charles Chihara, Stewart Shapiro, Mark Balaguer and Øystein Linnebo reveals not only substantial disagreement about the correct answer to this question, but also what the question is in the first place. For example, Chihara presents a nominalist interpretation of mathematics that accepts mathematical practice without positing the existence of any abstract objects. He offers a "structural account" of mathematics according to which mathematical axioms and theorems are not presented as true, but only as characterizing a model in which they are true. A benefit of this approach "is that we can avoid having to justify any analysis of what the assertions of the mathematical theories truly mean" (p. 144) and we can explain why "mathematical practice is simply not concerned with reference" (p. 154). By contrast, Balaguer insists that the main aim of a philosopher of mathematics is to construct a semantic theory which "is a presumably empirical theory about what certain expressions mean (or refer to) in ordinary discourse" (p. 181). This leads Balaguer to a fairly comprehensive survey of the different ways in which nominalists and platonists have tried to provide a semantic interpretation of mathematical language, although of course Chihara's non-semantic proposal is missing. In the end Balaguer argues for the somewhat counterintuitive conclusion that there is exactly one defensible version of nominalism and platonism, but that "there is actually no fact of matter" (p. 201) about which is correct.

In different ways, Shapiro and Linnebo develop non-standard interpretations of the nature of mathematical objects like the natural numbers 1, 2, 3, ... Shapiro, perhaps the most influential philosopher of mathematics working today, defends an "ante rem structuralist" interpretation of mathematics which posits the existence of abstract structures, like the natural number structure, that are metaphysically prior to the positions occurring in the structure, like the particular natural numbers (p. 173). A strong consideration in favor of this kind of structuralism, as against ordinary platonism, is that it can account for the way in which mathematicians seem to treat different set-theoretic structures as "the" natural numbers. Linnebo arrives at a very different understanding of the natural numbers by focusing on the family of systems of numerals which are adequate to represent the natural numbers. Linnebo argues first that we must permit significant differences in our understanding of reference to abstract objects and physical objects. For the natural numbers, this reference involves systems of numerals. The parallels between the numerals and the numbers is so tight that "whenever a natural number n possesses some arithmetical property, its doing so is inherited from the fact that the numerals that present n possess some related property" (p. 215). This leads Linnebo to conclude that even though there is still a sense in which numbers exist and numerals refer to numbers, the traditional problems for platonism can be overcome based on his understanding of the numeral-number link.

A highlight of the volume also comes in part three with the mathematician Barry Mazur's essay "When is One Thing Equal to Some Other Thing?". He offers one of the most accessible introductions that I am aware of to the basic idea that category theory can be used to characterize a mathematical domain independently of any particular choice for an underlying set theory. This sets the stage for an explanation of the tendency to treat isomorphic entities as equal. This point of view is illustrated using a characterization of the natural numbers as an initial object of what Mazur presents as the Peano category. This has advantages over a traditional presentation of the natural numbers as "it isolates, as Peano himself had done, the fundamental role of mere *succession* in the formulation of the natural numbers" (p. 232). This sort of approach to a mathematical domain is presented as a useful intermediate approach between an extreme "bureau-of-standards kind of definition" which arbitrarily selects one progression as the natural numbers and the alternative "Fregean universal quantification approach" (p. 232) which tries to define the natural numbers by quantifying over absolutely everything.

The final part of the volume has five essays which aim to either treat mathematics as a whole or else investigate the relationship between mathematics and its applications. The mathematician R. S. D. Thomas argues that mathematics is best understood "as sitting at the extreme of a spectrum of sciences" (p. 249) where the special focus of mathematics is the relations between things as opposed to their non-relational intrinsic nature. Thomas bases his proposal partly on a historical discussion of how mathematics has changed over time, and also on a consideration of the priorities of some current areas of mathematical research. Keith Devlin's aim in his essay is instead to investigate "What Will Count as Mathematics in 2100?". He argues that mathematics will be extended to incorporate patterns that are found outside the traditional domains of mathematics and its application. These include the notion of utility, Bayesian confirmation theory, theories of linguistic communication and financial economics applications such as the Black-Scholes model for option pricing. All such areas are partly mathematical now, but Devlin claims that further study will isolate patterns that are sufficient to expand mathematics to cover new territory.

The philosopher Alan Hájek considers one of these areas with his survey of probability theory and its philosophical interpretation. Hájek helpfully distinguishes debates about the different axioms of formal probability theory from the more philosophical issue of the viable interpretations of probability theory. For the axioms, Hájek reviews the problems concerning the popular approach that takes unconditional probability as basic and conditional probability as defined. Five different interpretations of probability are considered, including the logical interpretation developed by Carnap and the more subjectivist interpretation associated with De Finetti and Ramsey.

The more general problem of the applicability of mathematics is analyzed by the philosopher Mark Steiner. Steiner contrasts the logical problem of making sense of how mathematics contributes to scientific reasoning with the empirical problem of relating different mathematical concepts to different underlying physical regularities. Focusing mainly on the empirical problem, Steiner argues that some ways in which mathematics is currently used in science defy any straightforward explanation. In particular, he considers the way in which "isotopic spin" is represented using vectors in complex vector spaces and manipulated using the $SU(2)$ group. After explaining this treatment for some calculations involving particles and their isotopic spin, he suggests that they may involve the "Pythagorean Principle" that "At the deepest level of description, physical systems which are mathematically equivalent are physically equivalent – and thus one can be transformed into the other" (p. 320). Here, then, is a link from the application of mathematics to deeper philosophical mysteries about the nature of the physical world.

Finally, the mathematics education professor Guershon Harel explains how a broad conception of what mathematics amounts to can inform a program for improving mathematics education. Harel claims that we should not only focus on the products, or "ways of understanding", of mathematics, but also the process or "ways of thinking", that students and mathematicians employ to arrive at these products. He outlines an approach to mathematics education that emphasizes the intellectual needs of students and relates these needs to the sorts of assignments that students should be asked to complete.

3 Opinion

Hopefully these brief summaries suggest how the editors have sought to link philosophy of mathematics more closely with the interests of mathematicians. There is certainly a need for more engagement between mathematics and the philosophy of mathematics and I believe that this volume marks a productive first step in this direction. It is worth briefly asking, though, what barriers there are to philosophy-mathematics interaction and whether this volume will do much to overcome them. As I have already emphasized, philosophers and mathematicians tend to approach a philosophical topic with different priorities. The mathematicians in this volume often emphasize examples and exciting developments within mathematics, while the philosophers spend most of their energy clarifying concepts and criticizing the arguments of other philosophers. When taken to extremes either approach can frustrate the members of another discipline. Philosophers rightly ask mathematicians to clarify and argue for their positions, while a mathematician may become impatient with endless reflection and debate. A related barrier is the different backgrounds that most philosophers and mathematicians have. Philosophers are typically trained through the careful study of their predecessors and are taught to seek out objections and counterexamples. While most philosophers of mathematics have an excellent understanding of foundational areas of mathematics like logic and set theory, for obvious reasons few have reached a level of specialization in any other area of mathematics. By contrast, most mathematicians will not have much of a background in philosophy and will be tempted to appeal to the most interesting examples from their own mathematics even if they are not accessible to philosophers, let alone many other mathematicians. I am happy to report that most of the philosophical and mathematical discussion in this volume should be fairly accessible to everyone, but this probably happened only because the editors were looking out for complexities that might put off the average reader. Finally, it would be a bit naive to ignore the substantial professional barriers that stand in the way of any substantial philosophy-mathematics collaboration. To put it bluntly, nobody should try to get tenure by publishing for a community outside their home discipline. That said, it is encouraging to see philosophers and mathematicians at least trying to engage each other's interests and I hope these efforts will be continued and expanded in the coming years.