Mathematical Explanation Requires Mathematical Truth[*]

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I. An explanatory argument for mathematical platonism

One way to appreciate the challenge that mathematical explanation poses for the philosophy of science is to highlight a contrast between two sorts of naturalism. Quine advocated a thoroughgoing naturalism that counsels that philosophers should only criticize science using the tools and standards found within science. Quinean naturalism thus rejects any foundationalist first philosophy that would presume to dictate what knowledge is or how science should be done. In his book *Word and Object* Quine deploys a famous simile from Neurath as one of his epigraphs to drive this point home: "We are like sailors who must rebuild their ship on the open sea, never able to dismantle it in dry-dock and to reconstruct it there out of the best materials" (Quine 1960). Neurath anticipated Quine's naturalism in a number of respects, but they ultimately clashed on the issue of metaphysics. For Neurath followed his simile with the claim that "Only the metaphysical elements can be allowed to vanish without trace" (Neurath 1932, 201). In a properly naturalized science, Neurath supposed, metaphysical claims would be removed because they contributed nothing to the success of science.

Quine disagreed with Neurath on the status of metaphysics. For Quine, metaphysical commitments do earn their keep through their contribution to the overall success of science. This point is explicit in "Two Dogmas of Empiricism":

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"Ontological questions … are on a par with questions of natural science" (Quine 2004, 52-53). Whether or not we should believe in neutrinos is settled by determining which theory best accommodates our interactions with the world. Similarly, for Quine, questions about the existence of platonic mathematical entities like numbers and sets should be answered by considering our best scientific theory. If this theory makes use of statements that imply the existence of these entities, then we should believe that they exist. Metaphysical elements may turn out to be central to the success of science, and the traditional metaphysical questions that Neurath had sought to expunge are revived in a new form.

Quine's argument for the existence of platonic mathematical entities can be resolved into two parts. First, there is Quine's criterion for ontological commitment. Quine claims that a scientific theory is a collection of statements, but that these statements must first be regimented in a definite way if one is to assess what their truth requires of the world. For Quine this regimentation uses only first-order logic. Once this process is complete, there is a determinate test that decides if that theory posits the existence of entities of kind F. One asks if the statements of the theory entail "There are Fs" (or in symbols: $(\exists x)$ (Fx)).

Regimentation of scientific activity leads to a collection of conflicting theories that a scientist or philosopher could assent to. The second part of Quine's argument considers the various theoretical virtues that we wish a theory to have. These virtues are determined by an examination of how science is done. Quine's naturalism means that it is only scientific practice that can settle which theory is the best. At one point he proposed five virtues: conservatism, generality, simplicity, refutability and modesty (Quine and Ullian 1978). We aim for general theories that make sense of many phenomena using the simplest viable means. But the generality of the theory should not sacrifice the testability and refutability of its central claims in light of future experiments. Similarly, our theories should be modest in that they should not make unwarranted leaps into speculation. Quine also emphasized, under the heading of "conservatism", the value of retaining the beliefs that one begins with.

Quine's considered position, then, is that the regimented theory that maximizes the theoretical virtues will posit the existence of platonic mathematical entities. Arguments of this sort have come to be known as indispensability arguments for mathematical platonism. The main question for these arguments is whether it would be better to opt for a revised theory that limited its commitments to platonic entities, or perhaps eliminated them entirely. This sort of revised theory might be less simple, less general and less conservative than the platonic theory that Quine preferred. But it would seem to score well on the virtues of modesty and refutability. Platonic entities are a seemingly strange addition to our metaphysics, and their features elude any direct experimental testing.

The upshot of this debate is that there is no simple recipe for arriving at the best theory. Quinean platonists embrace a scientific argument for the existence of mathematical entities that relies on a certain list of theoretical virtues as well as a criterion for ontological commitment. Nominalists, who deny the existence of any abstract entities, thus have two points of attack. They may revise or deny Quine's test for the ontological commitments of a scientific theory. Or the nominalist may question the application of the Quinean theoretical virtues. The best theory, all things considered, may not require any abstract objects.[1]

This standoff between Quinean platonists and nominalists has motivated a new debate about inference to the best explanation (IBE). Many scientists take the explanatory power of a proposed theory to be part of the evidence that the theory is true. The Big Bang theory is said to provide the best explanation for the observed pattern of background radiation. The theory that a bolide (comet or asteroid) hit the Earth near Mexico is presented as the best explanation for the extinction of the dinosaurs, which is in turn accepted as the best explanation for the observed distribution of dinosaur fossils. An explanatory indispensability argument presents these explanatory considerations as a sufficient reason to accept the existence of platonic mathematical entities. For it appears that many scientific theories make essential use of mathematical entities when they explain observed phenomena. These explanations are the best available because they exhibit various explanatory virtues. These virtues include the generality and simplicity that we have seen already in connection with theories. But now with explanations we want an explanation to explain as much as possible using the simplest means.

The advocate of the explanatory indispensability argument points to specific explanations that rely on mathematical entities. One example that has received

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[1] See Colyvan (2015) for a very thorough survey of these debates. Azzouni (2004) questions Quine's criterion for ontological commitment. Field (1989) and Maddy (1998) contest the claim that theoretical virtues favor a platonist interpretation of our best scientific theories.

extensive discussion is the geometric structure of the honeycombs that bees construct to store their honey. A two-dimensional cross-section of these honeycombs shows a hexagonal tiling pattern, and scientists seek an explanation of this trait. The mathematical explanation is that this trait is an evolutionary adaptation of the bees that allows them to cover the most area using the minimum amount of wax. This is the fittest trait because it does the most with a costly resource. And this explanation is said to be the best explanation because it exhibits virtues like generality and simplicity.

Varro's *On Agriculture* from 36 BC presents this explanation as superior to one that cites the number of legs that bees have: "Does not the chamber in the comb have six angles, the same number as the bee has feet? The geometricians prove that this hexagon inscribed in a circular figure encloses the greatest amount of space" (Cato & Varro 1934, 501, noted by Hales 2000, 448). The mathematical claim at the heart of the explanation is that hexagons maximize the ratio between the area covered and the length of the edges used in the tiling. A proof of this theorem for the special case of regular polygons appears in Pappus' *Mathematical Collection* from around 340 AD, but was probably known much earlier (Nahin 2007, Appendix C). The theorem was proved for all polygons only in 1999 by Thomas Hales. When regimented, it entails the existence of mathematical entities such as hexagons and ratios. The defender of the explanatory indispensability argument concludes that our best science provides adequate support for the existence of mathematical entities. Scientists wish to explain observed phenomena, and to do so they must accept the existence of the Big Bang, a bolide hitting the Earth, hexagons and ratios.

There is thus supposed to be no principled way to be a defender of IBE and also a nominalist.[2]

II. Explanation and truth

IBE takes the explanatory power of a proposed explanation as evidence that the proposed explanation is in fact true. The Quinean platonist may be content to motivate this connection through an appeal to scientific practice: scientists, we are told, do use explanatory power as part of their evidence. If so, any naturalist should take this form of inference as valid. This argument ignores the critical attitude that is implicit in naturalism. For the naturalist can and should reject a given instance of IBE if it clashes with our scientific methods and standards. Many scientists seem uncomfortable with an unrestricted form of IBE that takes being the best available explanation as the only factor in theory choice. An example that has been emphasized by Maddy concerns the existence of atoms (Maddy 1998). In the first half of the nineteenth century Dalton proposed the existence of atoms as part of his explanation for the proportions found in various chemical reactions. However, this attempted IBE argument for the existence of atoms was unconvincing for most scientists. It was only at the beginning of the twentieth century, with the work of Perrin in particular, that a persuasive IBE argument was found. This suggests that a proposed explanation must meet additional restrictions beyond its potential explanatory power before it is accepted by the scientific community. Once we

[2] Baker (2009) and Saatsi (2011) discuss many of these vexing issues. Saatsi (2016) and Baker (forthcoming) are two important recent contributions to the debate.

consider what sort of restrictions should be in place, then it becomes reasonable to worry that a novel ontological commitment is excessive.[3]

The major step in any IBE is the consideration of how the correctness of the proposed explanation would account for the phenomenon in question. In the extinction of the dinosaurs case, it must be clear how a bolide hitting the Earth at that place and time would make sense of the later extinction of the dinosaurs. A similar point holds for the honeycomb case. If we consider the possibility that a hexagon really is the best basic shape with which to build a honeycomb, then this should illuminate the presence of that shape in the bee's honeycombs. To say that these proposed explanations are the best is to say that, of all the available proposals, these ones exhibit the explanatory virtues to the highest degree. These virtues do not entail that the best explanation is true. For all we can tell, the correct explanation of these phenomena might not have been formulated for our consideration. Another possibility is that the correct explanation has been formulated, and yet as presented it fails to show the explanatory virtues. The defender of IBE can admit the fallibility of their inference. However, there must be something special about uses of IBE that lead us astray. Otherwise, it would be unscientific to persist in using this form of ampliative inference.

Like any other kind of evidence, the explanatory evidence that drives these inferences can be degraded. Two sorts of scenarios are worth considering. In the

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[3] See Mayo (1996) for another discussion of the atoms case. I have provided a preliminary comparison of the atoms case with the mathematics case in Pincock (2012), ch. 10.

first scenario, we receive independent evidence that a central claim of the best explanation is in fact false. In the second kind of situation, the original best explanation is shown to be inadequate through the formulation of an even better explanation. In both cases, a defender of IBE should advise a shift away from the original claims.

To see how such cases would work, consider again the bolide explanation for the extinction of the dinosaurs. Suppose that the only evidence for the existence of this bolide impact is the way that claims about the bolide contributed to the best explanation of the extinction. Scientists might arrive at independent evidence that indicates that no bolide could have struck the Earth during the relevant time period. If this evidence was strong enough, then it would be prudent to withdraw our commitment to that bolide, even if it left us with no explanation of the extinction of the dinosaurs. In a second kind of case, suppose that additional investigations led to a better explanation of the extinction in terms of volcanic activity rather than a bolide. This might involve a more detailed understanding of what drives volcanic activity along with new indications of a rise in volcanic activity right before the dinosaurs went extinct. Again, as in the first case, the rational thing for scientists to do would be to withdraw their belief in the bolide, and opt for the volcano explanation.

The honeycomb explanation has a different character than the bolide explanation, but the same points apply concerning the defeasibility of explanatory evidence. For the first kind of case, we need only suppose that mathematicians find a flaw in Hales' recent proof or even the ancient proof presented by Pappus. This could lead to a new theorem that some other polygon maximized the ratio between its area and the length of its perimeter. If the support for this new theorem was strong enough, then the original explanation of the hexagonal shape of the bees' honeycomb would have to be rejected. Also, if the only support for the existence of hexagons and ratios was this explanatory evidence, then the prudent scientist would withdraw belief in those platonic entities. A slightly different result obtains in the other kind of scenario. Suppose an innovative scientist proposed a new explanation of the hexagonal shape of the honeycomb cells that tied that shape to the fact that bees have six legs. For us it is a mere coincidence that the cells have six sides and the bees have six legs. So this proposal scores very poorly as a proposed explanation as there is no illuminating connection between these numbers. But some future scientific development could fill in this connection, and come to surpass the current geometric explanation in explanatory power. If this happened, then scientists would not have a reason to accept the hexagons and ratios of the original explanation. They would instead opt for a commitment to the numbers that are central to the new explanation. That is, they would shift from a belief in one sort of platonic entity to another sort of platonic entity.[4]

Thinking through these sorts of scenarios highlights the strong link between genuine explanation and truth. IBE is a form of ampliative inference that, like all

[4] Some platonic entities may be more easily replaced by nominalist surrogates. For example, a commitment to finitely many natural numbers may be reinterpreted in terms of concrete objects, while a commitment to infinitely many natural numbers is much harder to reinterpret. See Pincock 2012, ch. 10 for some additional discussion.

inference, aims at truth. To properly carry out the inference, we do not need to know that the conclusion we are drawing is in fact true. But the inference should be corrected and reconsidered as new evidence comes in. If we have accepted claim C due to its presence in the best available explanation, then we should reconsider C if it turns out that C is not part of the best available explanation. In one scenario, we receive independent evidence that C is false. If this new evidence is strong enough, then we should abandon our commitment to C and also to the explanation that we previously accepted. This may leave us with no explanation of the phenomenon in question. In a second scenario a better explanation is proposed and this new explanation turns out to lack C. In this case, we should switch to this better explanation and drop our acceptance of C.

This pattern of acceptance and rejection only makes sense if we suppose that genuine explanations are composed of true claims. For the special case of mathematical explanations of physical phenomena, mathematical explanation requires mathematical truth. It turns out that there are accounts of scientific explanation that do not require truth. For example, van Fraassen's pragmatic theory of explanation insists that explanations are answers to why-questions (van Fraassen 1980). Van Fraassen claims that we can use the theories that we accept to answer these why-questions even if we do not believe that these theories are true. Given the discussion of this section, it is unsurprising to find that van Fraassen rejects IBE (van Fraassen 1989). On his approach, there is no link between a genuine explanation and truth. And so there is no way to motivate IBE.

A defender of IBE should maintain that explanation requires truth. Many scientific realists go further than this. They suppose that IBE can be used to characterize the natures of the entities involved in their best explanations. However, in scientific uses of IBE, there is widespread caution in using explanatory considerations to settle contested questions. In the bolide case, even though the bolide hypothesis was accepted based on explanatory considerations, the nature of the bolide that hit the Earth remained subject to debate. As early as 1980, the presence of iridium in the geological record at the time of the dinosaur extinction was sufficient to convince many scientists that an impact of some sort caused the extinction. This explanation was judged to be better than its primary competitor, the volcanic activity hypothesis. It was only in 1991, when the Chicxulub crater was identified, that the size and composition of the bolide was ascertained with sufficient confidence. And it was only this additional information that narrowed down the nature of the bolide to an asteroid of a specific size (10 km wide). Debates continue concerning the significance of the timing of the asteroid's impact. While some argue that the impact would have caused the extinctions no matter when it occurred, others maintain that an earlier impact would not have led to the extinction of the dinosaurs (Schulte et al., 2010, Brusatte 2015, Brusatte et al., 2015).

We can make sense of the change from the initial acceptance of the truth of some claim to the later acceptance of the existence of a new type of entity if we suppose that IBE is essentially eliminative. On this approach, when a scientist accepts a hypothesis using IBE they are using explanatory considerations to eliminate competitors. For example, in 1980 the iridium was used to eliminate the volcano hypothesis in favor of the impact hypothesis. But this choice in favor of the impact hypothesis did not yet settle the character of the impact, and so scientists remained agnostic on these additional details. It was only when a specific crater was identified in 1991 that a new IBE argument was possible that pinned down the character of the body that impacted the Earth. This additional information could only be explained by a certain kind of asteroid, and so the hypotheses concerning other sorts of bolides were eliminated from consideration.

These reflections on IBE raise some problems for the Quinean platonist. Even if one accepts the explanation-truth link, one needs independent motivation to draw any conclusions about the natures of the mathematical entities from their putative role in these scientific explanations. The problem is that the mathematical entities figure in these explanations solely in terms of their abstract structure. In the bees case, the theorem requires only that a certain ratio hold between areas and perimeters. It does not require anything of the geometric entities themselves. Similarly, a mathematical explanation that makes use of arithmetic will turn only on the structure of the natural numbers and their structural relationships. This will not pin down the character of the natural numbers. The explanation has its virtues, and the mathematical claims that figure in the explanation may be true, no matter what additional intrinsic features the mathematical entities may possess. Unlike the asteroid case, there is no forthcoming information that would allow scientists to eliminate some of the hypotheses concerning the natures of these entities, and so support a single remaining existential hypothesis. I conclude that it is illegitimate for the advocate of the explanatory indispensability argument to use their argument to say anything more than that some mathematical claims are true.[5]

III. Truth and fiction

The main alternative to the claim that mathematical explanation requires mathematical truth is known as fictionalism. Fictionalists propose a link between the mathematical claims that we accept and fiction. Consider, for example, Pappus' theorem that hexagons are the regular polygons that cover a given area using the shortest perimeter. A traditional platonist insists that this is a true claim about geometric entities and cites the proof as conclusive evidence for this truth. A Quinean platonist worries that the axioms used to obtain the proof may be false, and so rejects traditional platonism. But the Quinean platonist would accept the theorem (and perhaps also the axioms) if that package of claims made important contributions to scientific explanation. A fictionalist remains unconvinced. On their approach, false claims may be used in science, even in explanations, and even when we think they are false. This is because these claims appear in a coherent collection of fictional claims. Fictional claims are true in a given fiction, but may be false of the real world or "outside" the fiction. The thought is that fictional claims can explain features of the real world. Their falsity is no barrier to their explanatory power.

No matter how the collection of fictional claims is specified, the fictionalist faces the challenge of clarifying how claims that are false can figure in a genuine explanation. One version of this challenged has been pressed by Colyvan in his

[5] This objection is somewhat similar to Sober (1993). However, Sober goes further and argues that not even mathematical truth is legitimated by these considerations.

discussion of “easy road” nominalism. An easy road nominalist uses mathematics in scientific explanations while denying that these mathematical claims are true. This differs from a “hard road” nominalism that tries to formulate non-mathematical versions of the scientific explanations that we accept. The hard road nominalist can offer their reformulations for comparison with the original mathematical explanations and presumably try to make the case that the nominalist explanations are superior. The easy road nominalist is not able to do this because they work with the original explanations. What they add is some kind of qualified acceptance of the truth of the claims that appear in these explanations. The purely mathematical claims are relegated to the purely fictional, and so are said to explain even though they are false.

Colyvan’s challenge is quite simple: “when some piece of language is delivering an explanation, either that piece of language must be interpreted literally or the non-literal reading of the language in question stands proxy for the real explanation” (Colyvan 2010, 300). The fictionalist proposes a fictional or non-literal interpretation of the claims found in the explanation and appears to deny that there is any additional “real” explanation backing it up. However, if this is all the fictionalist says, then they are not able to vindicate some of the core features of an explanation. In the dinosaurs case, for example, the asteroid proposal explains the extinction by linking the asteroid event with the later extinction. If we accept the explanation, then we take ourselves to have a reason for the extinction: the extinction occurred because that asteroid hit the Earth. It is not an explanation to be told that an asteroid hit the Earth, but that this should not be taken literally as it is only part of a collection of fictional claims. Colyvan's challenge, then, is to supplement this bare proposal with some more substantial account of how fictional claims may explain.

One suggestion has been made by Elgin (Elgin 2009). She notes that fictional entities may exemplify properties that are also instantiated in the real world. So we may compare a purely fictional scenario with an actual one, and provide the explanation through this sort of comparison. This proposal seems to work best with fictional characters such as Sherlock Holmes. Holmes' superior intelligence generates various traits such as impatience and an inability to show affection for his close friend Watson. The fiction thus illustrates a link between certain kinds of character traits. We see how intelligence can bring about impatience with others. When we see these features instantiated in the real world in our colleagues, then the fiction can be useful in explaining this connection. This colleague is impatient with us because she is so intelligent, just as with Holmes.

I maintain that this is not an adequate response to Colyvan's challenge. To see why, we can distinguish how one learns an explanation from the explanation itself. In the case just reviewed, it would be wrong to say that my colleague is impatient because Holmes is impatient, or that my colleague is impatient because Holmes is impatient due to his intelligence. My colleague is impatient because she is so intelligent. That is the explanation. I may learn the explanation by reading the literary fiction in the sense that I become aware of a potential explanatory link between these traits by these means. But the explanation itself makes no use of the fiction. If there is a link between these traits, then my colleague's impatience is due to her intelligence. This is a true explanatory claim.

A different proposal has been defended by Leng (Leng 2010, 2012). Leng's view can be seen as an attempt to extend the point made at the end of section II about the limited role of the natures of mathematical entities in mathematical explanations. If we suppose that these explanations require only an abstract structural relation between mathematics and the real world, then it might seem like this relation could obtain between the imagined subjects of fictional claims and the real world. On this view, one side of this relation does not exist and the claims we make about it are all false. Nevertheless, it is in virtue of this relation that a real world fact is explained.

I claim that Leng's structural proposal is also not an adequate response to Colyvan's challenge. Consider again the honeycomb case. The explanation claims that the fittest trait for the bees is to construct their honeycombs in hexagons. Part of the explanation makes a link between these shapes and the minimal use of wax. On the platonist reading, the mathematical theorem is true and it is the reason that this trait is the fittest. The fictionalist cannot say this. Leng must say that the mathematical theorem is true in the fiction, and its being true in the fiction is the reason that this biological trait is the fittest. The link between this truth in the fiction and the fitness of the trait is that there is an abstract structural relation between these fictional entities and the bees.

Just like Elgin, it remains quite unclear what sort of explanation this is. Any attempt at clarification risks offering a non-mathematical explanation or else failing to offer an explanation at all. For example, Leng could say that the fictional truth of the mathematical theorem entails a family of real-world counterfactuals: if the bees had built the honeycombs using different shapes, they would have used more wax. These counterfactuals may very well be true of the real world. It remains to be seen, though, whether it is these counterfactuals that figure in the explanation, or whether it is the link between the mathematics and the counterfactuals that explains. If it is the former, then the mathematics is merely helping us to learn the explanation. The explanation is not itself properly mathematical, and Leng is engaged in some kind of "hard road" project of removing mathematics from scientific explanations. This is not her intention. On the latter option, we again confront a mysterious link between a fictional claim and a real world claim. The connection between the fictional claim and the counterfactual is no clearer than the earlier link to the fitness of some biological trait.

I conclude that fictionalists who offer mathematical explanations while denying mathematical truth must do better if they are to respond to Colyvan's challenge. If explanation requires truth, then possessing an explanation requires indicating what is true. A fictionalist invokes a family of claims which are deemed to be false. The fictionalist thus cannot claim to have presented an explanation until they indicate where the true lies.

IV. Naturalizing the a priori

The proposal outlined here for the philosophical significance of mathematical explanations in science might seem dangerously unstable. A scientific realist who accepts the legitimacy of IBE is able to use these mathematical explanations to justify some purely mathematical truths. This justification is available because of the link between explanation and truth as well as the special explanatory power that mathematical truths provide. However, the way that mathematical truths explain is via their abstract structure. At the end of section II I argued that this means that there is no IBE argument that will characterize the natures of mathematical objects. The problem is how one can accept the truth of some claim about some mathematical domain without also taking a position on the intrinsic natures of these objects.

As I see it, there are only two sorts of solutions to this problem. The Quinean naturalist permits no other means of justifying claims. So they should admit that they cannot know the natures of mathematical objects and defend a kind of agnostic position about their character. If the Quinean is truly neutral on these issues, then they cannot draw a platonist conclusion.[6] The platonist position requires the existence of abstract objects which lack causal powers and reside outside of space and time. But these features are not implicated in the mathematical explanations deployed in science. All that is exploited are structural relations, and these can obtain if mathematical entities are construed in nominalistic terms. The Quinean naturalist has no way of ruling these interpretations out of bounds.

The second sort of solution is to allow some means for justifying one specific interpretation of purely mathematical claims. Quine tried to do this using his theoretical virtues, but our discussion in section I highlighted how unconvincing this is to the determined nominalist. One source of justification associated with

[6] For additional discussion of Quine's views on ontology, see Hylton 2007, ch. 9.

traditional platonism is a priori justification. That is, we may have the capacity to support certain claims about mathematical entities independently of whatever course of experience we have had. If this capacity exists, then an IBE argument for the truth of some mathematical claims could be supplemented with an a priori argument concerning the natures of the subject-matter of these claims. On one view, for example, mathematics is about abstract structures. This platonist or “ante rem” structuralism is partly motivated by the role of mathematics in science, but it is also supported by the way pure mathematics has developed over time (Shapiro 1997). It remains to be seen what claims about mathematical structures must be a priori justified in order for ante rem structuralism to be vindicated.

By way of conclusion, we can briefly consider how the admission of a priori modes of justification can cohere with a naturalistic approach to mathematics more generally. Casullo has argued for two controversial claims about a priori justification that bear on this issue (Casullo 2003). First, a priori justification need not confer certainty. Justification in general can obtain independently of certainty, and so it is unfair to require certainty for the special case of a priori justification. Second, the existence of a priori justification is best shown through an empirical investigation of human agents. A priori arguments for the existence of a priori justification are not convincing and fail to make contact with the reliability of human reasoning. If a priori sources are to be found, then we need empirical studies of how humans reason that focus on the reliability of this form of reasoning. A promising place to begin would be with an investigation of reasoning in those areas that have been traditionally linked to the a priori such as logic, mathematics and ethics.

A naturalist should be open to the investigation and discovery of these a priori sources because the naturalist is willing to let scientific investigations shape their philosophical commitments. In this respect, both Quine and Neurath forced an overly dogmatic frame on their naturalistic approach to the philosophy of science. Neurath can be seen to err in presupposing that all traces of metaphysics would vanish from the properly streamlined ship of science. Quine mistakenly supposed that all justification is empirical and tied to the confrontation between whole scientific theories and experience. Both thus conspired to block the possibility that a source of a priori justification would be sufficient to characterize the metaphysical natures of abstract objects such as mathematical objects. The true legacy of investigations into mathematical explanations in science may thus turn out to be a revised, more liberal form of naturalism that avoids "first philosophy" while also leaving open a richer form of interaction between mathematics and science.

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