A New Perspective on the Problem of Applying Mathematics†

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1. Introduction

In what follows I present what I take to be a fundamental problem concerning the application of mathematics in reasoning about the physical world. Despite the recent growth in interest in philosophical issues connected with applications, there has been little or no sustained discussion of what the options are for thinking about how mathematics relates to the physical world when it is applied.[1] I offer a 'new perspective' on the issue not by defending a novel account of how mathematics is applied, but rather by erecting a new framework for discussing the issue and by arguing for an account in terms of this new framework. I believe that this offers a new opportunity to review both the strengths and the weaknesses of an account of applications of mathematics in terms of mappings between the physical world and a mathematical domain.

2. The Problem

Imagine that it is October 5^{th}, 1957, the day after the U. S. S. R. launched Sputnik into orbit, and that we are part of a group of American scientists charged with the task of launching a similar satellite. In light of these disturbing developments we decide to scrap whatever preliminary projects we had for our satellite and return to square one. We take the specifications of the latest rocket engine and consider what velocity and height the rocket will reach before burnout. But the velocity and altitude that will be reached depends on the mass of the rocket, which decreases as it uses its fuel. We

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[1] See, for example, Steiner [1998], Balaguer [1998], and Colyvan [2001].

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begin with the assumption that the mass of the rocket carrying this satellite without fuel is 0.7×10^6 kg and that the mass of the rocket with fuel is 2.8×10^6 kg. As scientists, we know what to do next almost without thinking: we derive equations for the magnitudes that we are interested in and solve them. The equations will be things like:[2]

$$t = \frac{m_0 - m}{\alpha} \tag{1}$$

$$v = -gt + u \ln\left(\frac{m_0}{m}\right) \tag{2}$$

$$y = ut - \frac{1}{2}gt^2 - \frac{mu}{\alpha} \ln\left(\frac{m_0}{m}\right) \tag{3}$$

The equations are derived from basic equations of our scientific theory, along with certain assumptions like the conservation of linear momentum. Given our knowledge of mathematics we can solve these equations using the appropriate constants and we conclude that at burnout the rocket will be 100 km above the Earth and that it will have a velocity of 2.16×10^3 m/s away from the Earth.

We have all sorts of reasons to think that these particular equations were not used in actual reasoning about an American satellite. However, what I want to draw your attention to in this somewhat fanciful story is the natural way in which mathematics entered the picture. We started with a definite problem and brought in our scientific theory to help us solve the problem. At one stage we came up with certain equations, and we used our mathematical theories in this derivation and in the solutions to our equations. Nobody is likely to doubt that this was the right thing to do; it is, in fact, something we do all the time in science but also in ordinary life, *e.g.* when we decide not to buy something because we cannot afford it. Nevertheless, there is a philosophical problem lurking here, and that is what I would like to develop in this essay. The problem is quite simple to state, although like many philosophical problems we must be careful in deciding how to go about solving it. As philosophers, we want to know how and why this appeal to mathematics works. A moment ago when we were scientists, we began by wondering about rockets, satellites, and the Earth, and we came up with equations that govern rockets, satellites, and the Earth and solved these equations using mathematics. What is unsettling here is that our mathematical theories seem to be about things like numbers, functions, and operations like addition and multiplication. But why should a theory about these sorts of things be *relevant* to a rocket?[3] It is as if we started

[2] Here I follow Marion and Thornton [1995], §2.7. The specific numbers and equations are adapted from example 2.14, p. 93.

[3] See Shapiro [1983] for a similar formulation of the problem. Chapter eight of Shapiro [1998] contains further discussion of the issue.

talking about tables and chairs, and then moved to considering koalas and Eucalyptus trees, and finally returned to some verdict about tables and chairs. Mathematics seems to have a subject matter that is distinct from the physical world. The problem, then, is to say what connection there is between the physical world and mathematics that can explain the successful application of mathematics in scientific reasoning.

Here it might seem that we have stumbled upon a massive and seemingly intractable problem. For it looks as though I am asking for an account of how we can use mathematics to represent the physical world, and this might just be an instance of the general problem of how one thing can represent something else. However, I think we can sharpen the issue by focusing on the content or meaning of the statements that reflect our reasoning in these situations. We will have statements like 'The mass of the satellite is 100 kg', 'The gravitational acceleration is $9.81\frac{m}{s^2}$' and as our most complex example ' $y = ut - \frac{1}{2}gt^2 - \frac{mu}{\alpha}\ln(\frac{m_0}{m})$ (3)'. What all these statements share is the occurrence of what seems to be mathematical terms along with non-mathematical terms. I will call such statements 'mixed statements', and while there are certainly borderline cases between mixed and non-mixed statements, I will assume that we can all agree on some clear examples of mathematical and non-mathematical terms. If we can determine what these mixed statements mean then it seems that we will be in a much better position to answer the more general problem of how and why applications of mathematics are so important and successful in science.[4] This is the issue that I will discuss here: what are we saying when we utter mixed statements? And, following an influential Davidsonian tradition, I will take this question to be just asking what the truth conditions of mixed statements are.[5]

I will not be discussing the issue of idealization. When doing science we make all sorts of simplifications and idealizations, and this makes an account of applications much more difficult. For example, the above equations assume that the acceleration due to gravity remains constant as the rocket goes away from the Earth, while in fact we know that it is not a constant. In what follows, though, I will assume that the statements that I consider do not depend on, or assume, any idealizations. This assumption must, of course, be removed eventually, but to do so here would lead to further extended complications.

[4] That is, even if we have a satisfactory account of the meaning of mixed statements there might still be a further problem of explaining the success of applications of mathematics. But I take it that clarifying the meaning of mixed statements is a necessary first step in solving the more murky explanatory problem.

[5] See Davidson [1984]. I am well aware that many would reject this assumption by arguing that it biases the whole investigation. See, for example, Dummett's [1993a]. However, I must reserve for future work any discussion of the significance of different conceptions of meaning for this problem.

3. A New Perspective

When we look at the philosophy of mathematics since the 1960s, it turns out that most philosophers concentrate on questions about pure mathematics. There are, of course, exceptions. Stephan Körner, for example, puts applications of mathematics at the center of his *The Philosophy of Mathematics: An Introductory Essay* [1960], and uses the fact that mathematics is applied to highlight the shortcomings of the three traditional philosophies of mathematics—logicism, formalism and intuitionism. More often, though, the primary issue in the philosophy of mathematics is the debate between platonists and nominalists. Platonists believe that mathematics is the study of abstract objects like numbers and sets. The defining characteristics of abstract objects are usually that abstract objects are not in space and time, and that they do not engage in causal interactions with us. Nominalists offer a host of alternatives to platonism, from the proposal that mathematics is the study of empirical generalities to the claim that mathematics is the study of inscription tokens. All nominalists share the belief that mathematics is not about abstract objects. Typically these philosophers will set out their arguments for a platonist or nominalist philosophy of mathematics and only then explain how they can account for the application of mathematics.[6]

Nothing captures this unsatisfactory state of affairs better than the centrality of so called 'indispensability arguments' in the philosophy of mathematics. The most common kind of indispensability argument moves from the indispensable role of mathematics in our current best scientific theories to the conclusion that we must be platonists about mathematics. 'Indispensable' here is generally taken to mean that we cannot come up with an alternative scientific theory that does not use mathematics. If we cannot get rid of mathematics, so the argument goes, we must be platonists. If we can dispense with mathematics, then perhaps we can be nominalists.

The focus on indispensability arguments has, to my mind, obscured the general issue of how applications work because it has led philosophers to think of applications in different ways, depending on what their account of pure mathematics is. At its worst, indispensability arguments have led some platonists to think that applications are a problem only for the nominalist.[7] At the same time, nominalists often construct elaborate reworkings of our scientific theories, and then go on to argue that we *could* restrict ourselves to these new, non-mathematical, scientific theories.[8] Both sides seem to

[6] For a similar complaint see Bueno and French [1999], p. 38.

[7] As we will see shortly, Mark Steiner thinks applications can be explained quite easily by the platonist in terms of set membership.

[8] In his [1990] Charles Chihara clearly announces that he is not attempting to give an analysis of actual mathematics, including actual applied mathematics. Other nominalists are often less clear about what they intend their nominalist reconstructions of

have little to say about our problem of giving the truth conditions of mixed statements. Indispensability arguments, and the platonist-nominalist debate generally, occur at such a high level of generality that we often cannot tell what a given position entails for the truth conditions of mixed statements.

I propose that we think about the issue in a new way. I will consider accounts of applications in a way that is independent of accounts of pure mathematics. By taking up this perspective I believe that we can make real progress in thinking about the meaning of mixed statements. Let us return to a simple mixed statement: 'The satellite has a mass of 100 kg'. The statement seems to be about the satellite, the real number 100, the standard gram (assuming that there is such a thing), and some complicated relation between them. An initial division of accounts of applications can be made based on whether or not an account accepts this reading of mixed statements as claims about complex relations between mathematical and non-mathematical objects. An account that denies this will be called a 'no-relation' account. As we will see shortly there are two kinds of 'relation' accounts based on the kind of relation that an account claims exists between mathematical and physical domains.

No-relation accounts deny that any relation connects mathematical and physical domains when the mathematics is applied. The most prominent advocates of this approach to applications are fictionalists about pure mathematics. Fictionalists, such as Hartry Field, deny that there are any mathematical objects. Field has argued for his fictionalist philosophy of pure mathematics by considering how mathematics is applied.[9] He thinks that mathematics is dispensable from science. Field's idea is that all explanations should ideally be causal, and that if an explanation invokes entities that are not causally connected to any of the objects mentioned in the explanandum, then that explanation should be replaced by one which establishes more appropriate causal connections. As mathematical objects are not causally connected to the physical world, even if they exist, our physical explanations should not be couched in mathematical terms. Our question is: can Field's account of applications be made to work? That is, can we understand applications of mathematics in a way that stops short of requiring the existence of relations between physical and mathematical objects? On this proposal, a claim like 'The satellite is 100 kg in mass' could be true even if there were no relation between the satellite, the real number 100 and the standard gram. Its truth would depend, instead, only on properties of the satellite and its physical relations to other physical objects. However, as David Malament [1982] pointed out, in his review of

mathematics to accomplish.

[9] See Field [1980] and [1989].

Field's *Science Without Numbers*, many scientific theories are formulated in terms of phase spaces, where the points of these spaces are not anything like the supposedly concrete entities of physical space. Instead, each point in a given phase space represents a complete description of the system at issue. As the system will only take one route through the phase space, these are the only points in the space that can reasonably be said to exist. The problem for Field, though, is that many applications of mathematics to phase spaces require a comparison of paths. One can show, then, that Field cannot handle these applications of mathematics and preserve the spirit of his no-relation approach to applications.[10]

3.1 Internal Relations

Let me turn now to the second approach to applications of mathematics in my three-part classification scheme. This account accepts our original suggestion that mixed statements depend on a relation between mathematical and physical domains for their truth, and also adds a claim about the kind of relation that this is. The account claims that it is what I will call an 'internal relation'. Intuitively, an internal relation for an object a is a relation that a must stand in if it is to be that particular object. More precisely, a stands in an internal relation R to b just in case aRb's obtaining is involved in a's criteria of identity. A clear example of an internal relation is between a set and its members. Suppose a set, call it S, has b and c as members. A set that has different members will be a different set from S, and so we can say that S bears an internal relation to b, and also to c. This terminology is adapted from Bradley, Russell and Moore, although they often assume that if a relation is internal then its converse is also internal.[11] As my example makes clear, I am not making this assumption. It is part of set S's criteria of identity that it have b as a member, but it is not part of b's criteria of identity that it be a member of S.

What I will call an internal-relation solution claims that mixed statements of mathematics require that an internal relation obtain between the mathematical domain and the physical world. An important example of an internal relation solution is Frege's account of the natural and real numbers. Frege defined the natural numbers in terms of their paradigm application of counting, and he defined the real numbers in terms of their paradigm application of measuring.[12] For example, the natural number two is the extension that has in it all the first-level concepts that apply to two objects.

[10] For further details of this argument see Pincock [2002], ch. 4.

[11] Perhaps the clearest discussion of internal relations is to be found in Moore's [1922]. Moore does not make the assumption just alluded to. I cannot here discuss the connection between this definition of 'internal relation' and the definition according to which aRb is internal iff aRb's obtaining is equivalent to Fa and Gb for some properties F, G.

[12] See the articles in Demopoulos [1995] for further details.

In modern set-theoretic terms, the natural number two is a proper class that has as members all the sets that have two objects as members.[13] It is important to see here that the physical objects that the natural number two is applied to in counting will be members of a set that is in turn a member of the natural number two. Mathematical objects are tied directly to the physical situations that they are applied to by an internal relation. For Frege the relation was somewhat complicated: the natural number is an extension of a second-level concept, and it has in it first-level concepts that have physical objects in their extensions. In set-theoretic terms we can simplify this somewhat by talking of the transitive closure of a set. The transitive closure of a set S is a set S' that has as members all the members of S, all the members of the members of S, and so on. If we treat extensions as sets then whenever a natural number is applied in counting a physical object, this object must be in the transitive closure of the natural number. Note that a set bears an internal relation to all of the objects in its transitive closure. For example, consider the set $\{\{a, b\}, c\}$. Call it T. T's members are $\{a, b\}$ and c and the objects in its transitive closure are $\{a, b\}$, a, b, and c. If a set does not have a, b, and c in its transitive closure then it is not identical with T. In this way we see how Frege's account depends on an internal relation between mathematical objects and the physical world. The story for the real numbers is considerably more complicated, but an internal relation results in the end there too.

What seems to motivate Frege to offer such an account is the belief that it is not accidental that mathematics applies to the physical world. That is, we want an account of mathematics that *guarantees* that it will apply to the physical world.[14] At the same time, Frege went to great lengths to argue that an empiricist philosophy of mathematics achieved this guarantee only at the expense of sacrificing the necessity of mathematical truth. Dummett, in his perceptive article 'What is mathematics about?', has pointed out that these two *desiderata* are often in conflict:

> The difficulty about mathematical objects thus arises because [1] we want our mathematical theories to be pure in the sense of not depending for the existence of their objects on empirical reality, but yet [2] to satisfy axioms guaranteeing sufficiently many objects for any application that we may have occasion to make. (Dummett [1993b], pp. 437–438. My numbering.)[15]

I will refer to the choice between these two *desiderata* as 'Dummett's

[13] It is not clear whether or not Frege would have accepted this set-theoretic interpretation of his proposals. For the purposes of this paper, I am simply assuming that set-theoretic talk is sufficient to capture the sort of relation between mathematical objects and the physical world that Frege had in mind.

[14] This demand runs throughout Frege's writings but is especially explicit in §§14, 24, 104 of Frege [1980] and Vol. II, §157, of Frege [1893–1903].

[15] See also p. 445 and pp. 293-301 of Dummett [1991].

dilemma'. It seems that we must either choose to have mathematical objects stand in a direct relation to the physical world, and thereby sacrifice the necessity of mathematical truth, or accept a tenuous link between the mathematical and the physical worlds in order to preserve the independence of mathematical truth. According to Dummett, Frege's logicism was a brilliant attempt to bypass this dilemma by defending both the necessity of mathematical truth and the direct applicability of mathematics.

Though Frege's project failed because his system was inconsistent, it might still seem that his basic insight could be recaptured in a set-theoretic framework with physical objects as individuals. Perhaps the solution to Dummett's dilemma that Frege suggested could be saved by embedding it in a framework that avoids the contradictions that Frege's own logical system encountered. Each mathematical object would be a particular set or proper class, and it would have in its transitive closure the physical objects that were involved in its applications. The relation would be an internal one defined in terms of set membership, and so mathematics would apply directly, as it were, to the physical world. At the same time, though, the pure sets of the set-theoretic hierarchy, *i.e.* the sets that have no physical objects in their transitive closure, would guarantee the necessity of the mathematical truths, as their existence would be unaffected by the properties of the physical objects.

This is an initially attractive proposal, and it has in fact been suggested by Mark Steiner in his recent book *The Applicability of Mathematics as a Philosophical Problem*. At the beginning of his book Steiner is eager to solve what he calls 'the metaphysical problem' of applications. This problem arises 'from a gap between mathematics and the world, a gap that threatens to make mathematics irrelevant' (Steiner [1998], p. 19). Irrelevance is avoided by having physical objects be members of sets:

> To "apply" set theory to physics, one need only add special functions from physical to mathematical objects (such as the real numbers). Functions themselves can be sets (ordered pairs, in fact). As a result, modern—Fregean—logic shows that the only relation between a physical and a mathematical object we need recognize is that of set membership. And I take it that this relation poses no problems—over and beyond any problems connected with the actual existence of sets themselves (Steiner [1998], p. 23).

A direct connection is established between mathematical and physical objects, and so any worries about relevance should be removed.

Despite Steiner's appeal to Frege, his proposal is in fact ambiguous between an internal-relation solution and another kind of solution that I will consider later in this paper. The ambiguity arises from Steiner's omission of any clear definition of mathematical objects such as the natural numbers. For example, he says 'numbers characterize sets, not physical objects; while sets can contain, of course, physical bodies' (Steiner [1998], p. 22) but does not say whether or not numbers themselves are sets. Later, after citing Boolos's work on a second-order theory known as FA (for 'Frege Arithmetic'), he concludes that 'the only mathematical objects Frege needs for arithmetic are classes of equinumerous concepts' (Steiner [1998], p. 23). This strongly suggests that natural numbers are supposed to be sets (or proper classes). In what follows I will assume that Steiner is advocating what I have called an internal-relation solution, although, given only what he explicitly says, his account could be either an internal solution, a non-internal solution or perhaps even some hybrid.[16]

3.2 Possible Situations

But even though on this view mathematics applies automatically to the physical world and pure mathematics remains necessary, the internal relation leads to unanticipated consequences because it turns out that all objects involved in applications must be members of sets. To see the problem consider the original application of mathematics that we began with: designing a rocket to launch a satellite. Here we were reasoning about different possible ways of getting the satellite into orbit. Suppose we came up with a solution to (3) and it told us that the rocket would not be high enough at burnout to have the satellite enter a stable orbit around the Earth. We might still content ourselves with the observation that if we were able to build such a rocket, then it would burn out at some definite altitude, say 100 km. Here we are reasoning mathematically about possible situations. How can an internal-relation approach handle these applications of mathematics? I claim that the objects that we are considering here, the possible objects, must be individuals of the set theory. That is, if I am applying mathematics to describe a possible situation, then the relevant mathematical objects must bear an internal relation to the objects in this possible situation. As the framework that we are considering here is set theory, the possible objects must be members of sets and more specifically they must be in the transitive closure of the mathematical objects that are being applied.[17]

If this reasoning is correct, the internal-relation account is committed to a realist position about possible objects. There are two problems with this, though. First, why should we be realists about possible objects? It seems that only actual objects exist and the assumption that possible objects exist leads to a host of philosophical problems that may or may not be solvable. Second, and to my mind more importantly, accepting a given account of how mathematics is applied should not force a decision about the interpretation of modal logic. The two issues are separate. It does not seem right that

[16] See §5.3 for a brief discussion of hybrid approaches.

[17] The problem arises because we are considering counterfactual possibilities and not because these possibilities involve the future.

a solution to our problem about the meaning of mixed statements should resolve a complex issue about the interpretation of modal logic. As we will see later, some solutions to this problem leave a variety of options open for the interpretation of modal logic. This flexibility seems preferable, at least until the advocates of modal realism offer more conclusive arguments than they have to date.[18]

At this point one might respond that I have ignored one obvious development of the internal-relation account that would avoid realism about possible objects. The account could hold that the possible objects need not *actually be* members of the sets in question, but only that they *would be* members of these sets if the actual world turned out in a certain way. On this proposal it seems that an internal-relation account can avoid realism about possible objects. However, things are not so simple. To make my worry precise, consider an internal account of the natural number 1. It will have as members all the singletons of the actual physical objects. Now, suppose it is 1957 and I say 'There could have been exactly one American satellite in 1957'. On the current proposal I am saying that it is possible that the singleton of the satellite is a member of the natural number 1. But, note that it is not possible for the natural number 1 to have any new sets as members because it bears an internal relation to its members. There simply is no set that the internal-relation solution can hold fixed and consider as its members change. If the members change then there will be a new set. In order to get a definite natural number 1 we need a set theory that has all possible physical objects in it, and this is what leads to realism about possible objects. We want a single thing to be a natural number 1, but when we adopt an internal account of applications we are forced to bring in possible objects to preserve this definiteness across possible situations. Otherwise, claims that should come out true on any analysis will come out false.

One could avoid these problems by having different ones in different possible situations. However, this position only gives rise to further problems. Briefly, we will need an all-embracing theory about all these different possible ones and so realism about possible objects remains. When we claim, for example, that $1+1=2$ we are not just claiming that in the actual world it so happens that the one in this world added to itself yields the two in this world. The interpretation of this claim would rather have to be something like 'For all worlds w, $1_w + 1_w = 2_w$'. It seems clear that this quantification would require realism about possible objects. This realism follows from the assumptions that only existing objects can be members of sets and that a physical object must be a member of a set to be involved in an application

[18] For a compelling survey of and attack on arguments for modal realism see Chihara [1998].

of mathematics.[19]

I conclude that a wholly internal-relation solution can succeed only at a relatively high price. It remains to be seen what the alternatives are and whether these alternatives are any less costly.

4. The Structuralist Account

Let me turn, then, to the third kind of account of applications. On the external-relation account the mixed statements require for their truth the existence of an external relation between the mathematical domain and the physical situation. An external relation is any relation that is not internal, *i.e.* a relation that does not involve the criteria of identity of the mathematical objects. Before we even get into the details of this proposal, it is important to note how the external-relation account promises to resolve Dummett's dilemma. If the relations are genuinely external, then the connection between mathematics and the physical world will not affect the necessity of mathematical truth. That is, the mathematical objects and their mathematical properties will exist independently of the physical world and whatever changes it undergoes. At the same time, one would hope that an external-relation solution would supply enough of these external relations to ensure the applicability of mathematics to appropriate physical situations. This would satisfy Dummett's demand for a guarantee of the applicability of mathematics.

Let us turn to a concrete example. Consider the process of counting ordinary physical objects like apples on a table. We start with the natural number 1 and associate that with a given apple, and then proceed upwards through the natural-number series. For each natural number, we associate the natural number to a different apple. When we are done going through the apples, so that each apple is associated with exactly one natural number, we stop counting and say that the natural number that we have ended up with is the number of apples on the table. What is going on here? A natural thing to say is that there is a mapping of a specific kind from the apples on the table to an initial segment of the natural numbers. This mapping is called an isomorphism. Briefly, an isomorphism is a mapping that preserves cardinality and structure. Now, when I count the apples I am determining that there is an isomorphism from the apples to the natural numbers starting with 1. We can capture, then, the kind of external relation that is required by talking of mappings and their properties. Here we have a statement of the form 'There are n Fs' coming out true just in case there

[19] After this paper was largely completed I discovered this type of modal objection to Frege's account of the natural numbers in Hambourger [1977]. Hambourger does not draw any positive or more general conclusions from his objection, however. Also, he speaks of the 'Frege-Russell definition of number' in a way that overlooks the relevant differences between Frege's and Russell's proposals.

is an isomorphism from the Fs to an initial segment of the natural numbers ending with n.

This is the kind of external-relation approach which I will call 'the structuralist approach'. It is structuralist because it makes the truth of the mixed statements depend upon the existence of mappings that have the right properties, and these properties can be picked out in structural terms. There is also an account of pure mathematics known as structuralism.[20] Structuralists about pure mathematics maintain that the subject matter of mathematics is abstract structures, where these are distinguished from the abstract objects that traditional platonists posit. I want to be very clear that my structuralist account of applications is independent of a structuralist account of pure mathematics. Of course, the two are compatible as long as the structuralist about pure mathematics allows my relations to relate physical objects to the places or positions in her structures. In fact, although I cannot argue this point here, the structuralist account of applications is compatible with a number of accounts of pure mathematics, both platonist and nominalist. I say this mainly to avoid confusion and to make clear that, as far as I am concerned, the decision about what pure mathematics is about is a quite different issue from the correct account of applications of mathematics.

My structuralist proposal for applications, then, is that each mixed statement depends for its truth on the existence of a mapping with certain structural properties.[21] I have given an example of a mapping with fairly simple structural properties, but more complicated mappings are needed for other applications of mathematics. Two large issues remain. First, it is not yet clear what these mappings are, and when they will exist. Second, I have not shown how all, or even a large class, of applications can be handled in a structuralist fashion.

4.1 Properties

Consider a description of the length, mass and path of Sputnik around the Earth in 1957. Claims of length and mass are relatively easy to handle. They will require mappings from the satellite to the real numbers. Sputnik was .584 m in diameter and 83500 g in mass. According to the structuralist account 'Sputnik is .584 m in diameter' will be true just in case there is a mapping from the diameter of Sputnik to that real number, where this mapping is a homomorphism that, as we might say, preserves length. What is involved in this last qualification? The idea behind it is that the existence of just any mapping from Sputnik to the real numbers will not

[20] See, for example, Shapiro [1998] and Resnik [1997].

[21] Of course, many statements will involve more than one application of mathematics and so will require more than one mapping, but in general I will ignore this added complication in what follows.

ensure that Sputnik is .584 m in diameter. What we also need to add is that the mapping takes all meter sticks to 1, all objects that are half as long as a meter stick to 1/2, *etc*. These restrictions can be summarized using the results of the measurement theory developed by Suppes, Krantz, and others.[22]

In measurement theory the general question is: what assumptions must be made about a physical domain for it to be describable using a given branch of mathematics? Measurement theory approaches applications in a more or less structuralist way. So, the general question becomes: what sort of structure must a physical domain have if the right kind of mapping between the physical domain and a mathematical domain is to exist? Given these similarities with the structuralist approach it is not surprising that we can draw on the technical results of measurement theory in our description of the mappings needed for statements of applied mathematics to be true. However, I will adapt these results in a way that the practitioners of measurement theory may be not entirely happy with. For the structuralist account needs to place restrictions on the mappings themselves as part of the general strategy of saying that a certain statement will be true just in case a particular kind of mapping exists. This is a different goal from what is standardly investigated in measurement theory, where the focus is on the required structure of the physical situation. It is possible, though, to translate the results of measurement theory to suit the purposes of the structuralist account. The basic idea is simple. Instead of placing restrictions on the structure of the physical situation, as measurement theory does, we will transfer these restrictions to the mappings themselves. In this way we can define mappings of various types and then say that if a mapping, of a certain type, exists between a physical situation and a mathematical domain, then a given statement of applied mathematics will be true.

Before I turn to an example that will make this strategy clearer, I want to highlight another difference between measurement theory and the structuralist approach. Measurement theorists, for reasons that are not entirely clear, often take an operational approach to the physical properties of a system. For example, the relation $\succeq$ and the operation $\circ$ defined on some domain of massive objects are defined so that '$a \circ b = c$' means that if a is concatenated with b and placed on a scale with c, then the scale will balance:

> $a \circ b$ is an object in [the domain] A that is obtained by concatenating (or composing) a and b in some prescribed, ordered fashion. An example is weight measurement (...) in which the elements of A are material objects; $a \succeq b$ is established by placing a and b on the two pans of an equal-arm pan balance and observing which pan drops; and $a \circ b$ means that a and b are both placed

[22] See Scott and Suppes [1958], Krantz *et al.* [1971], Suppes *et al.* [1989], and Luce *et al.* [1990].

in the same pan with a beneath b. (Krantz *et al.* [1971], p. 72)

This operational approach leads to numerous problems, though, connected with the difficulties associated with carrying out these observations and the resulting idealizations needed for the desired results to be obtained.

In what follows I will drop this approach and adopt a generally realistic attitude towards physical properties of objects and systems. On my interpretation '$a \circ b = c$' involves the mass properties of a, b and c and makes a claim about properties of these properties. The equation will be true just in case object c has two non-overlapping and exhaustive parts that have the mass of a and the mass of b. Similarly, '$a \succeq b$' involves a relation between the mass properties of a and b. It will be true when an object with a's mass property has more than or the same mass as any object with b's mass property. In taking this attitude towards properties, then, physical properties are endowed with quite a rich structure.[23] There will be properties and properties of properties that may or may not be instantiated. By introducing such an array of properties it seems possible to avoid the difficulties associated with operationalism. For example, '$a \circ b = c$' on the operational approach seems undefined when the objects are too large for the pan mentioned. Still, a's and b's mass properties will or will not be related in the appropriate way to c's mass property, even if this is not something we can operationally (or by any means) determine.

Krantz *et al.* define a closed extensive structure as follows:

Let A be a nonempty set, $\succeq$ a binary relation on A, and $\circ$ a closed binary operation on A. The triple $\langle A, \succeq, \circ \rangle$ is a *closed extensive structure* iff the following four axioms are satisfied for all a, b, c, $d \in A$:

1. Weak order: $\langle A, \succeq \rangle$ is a weak order, *i.e.*, $\succeq$ is a reflexive, transitive, and connected relation.
2. Weak associativity: $a \circ (b \circ c) \sim (a \circ b) \circ c$.
3. Monotonicity: $a \succeq b$ iff $a \circ c \succeq b \circ c$ iff $c \circ a \succeq c \circ b$.
4. Archimedean: If $a \succ b$, then for any c, $d \in A$, there exists a positive integer n such that $na \circ c \succeq nb \circ d$, where na is defined inductively as: $1a = a$, $(n+1)a = na \circ a$. (Krantz *et al.* [1971], p. 73).[24]

Using this definition they show:

Let A be a nonempty set, $\succeq$ a binary relation on A, and $\circ$ a closed binary operation on A. Then $\langle A, \succeq, \circ \rangle$ is a closed extensive structure iff there exists a real-valued function ϕ on A ($\phi : A \rightarrow Re$) such that for all a, $b \in A$
(i) $a \succeq b$ iff $\phi(a) \geq \phi(b)$;
(ii) $\phi(a \circ b) = \phi(a) + \phi(b)$.
Another function ϕ' satisfies (i) and (ii) iff there exists $\alpha > 0$ such that $\phi' = \alpha\phi$ (Krantz *et al.* [1971], p. 74).

[23] For what appears to be a similar attitude see Stalnaker's brief discussion of 'families of properties which have a structure in common with the real numbers' in Stalnaker [1984], p. 9.

[24] Here $\succ$ and $\sim$ are defined in the obvious way in terms of $\succeq$.

Closed extensive structures will include lengths.

What we need for the structuralist account is a list of the conditions that a mapping must satisfy if it is to provide the appropriate analysis for claims involving lengths. By assuming a realistic attitude towards properties we can dispense with some of the clauses of the above definition, *e.g.* 1., 2., and 3. These clauses merely restate the essential properties of the mass or length properties that the realist countenances. For example, '$a \circ (b \circ c) \sim (a \circ b) \circ c$' must be true for any triple of mass properties.

This leaves the clauses of the theorem and clause four of the definition. A plausible attempt at giving the necessary and sufficient conditions for a mapping ϕ to satisfy the structuralist's requirements is:

For all actual objects a and b,
(i) $a \succeq b$ iff $\phi(a) \geq \phi(b)$; (4)
(ii) $\phi(a \circ b) = \phi(a) + \phi(b)$;
(iii) $a \sim nb$, where nb is defined inductively as above, iff $\phi(a) = n\phi(b)$.

A mapping satisfying (4) would take any object with a length property to a real number in such a way that the relations between the real numbers mirror the structure of the relations between the length properties. For example, suppose the universe consisted of one long rod. The rod and its various parts would have varying length properties. If ϕ satisfies clause (i) of (4), then it will take parts with different length properties to different real numbers such that if a's length property is greater than b's, then the real number associated with a will be greater than b's. The most significant clause is (iii), which ensures that ϕ respects the rich connections among the different lengths. It is not sufficient just to assign the right ordering of lengths, but the lengths must also be numbered in a way that reflects the fact that one length is twice another, *etc*.

The upshot of this discussion is the following analysis of a length claim such as 'The length of the rod is 2 m':

'The length of the rod is 2 m' is true iff there is a mapping ϕ that maps the standard meter to 1, the rod to 2 and that satisfies, for all actual objects a, b,
(i) $a \succeq b$ iff $\phi(a) \geq \phi(b)$;
(ii) $\phi(a \circ b) = \phi(a) + \phi(b)$;
(iii) $a \sim nb$, where nb is defined inductively as above, iff $\phi(a) = n\phi(b)$.

Let us call this sort of mapping length-in-meters homomorphisms. Now, the original claim 'Sputnik is .584 m in diameter' will be true just in case there is a length-in-meters homomorphism from the diameter of Sputnik to the real number .584.

Mass claims can be handled in a completely analogous way, where '$\succeq$' and '$\circ$' are spelled out in terms of mass properties and their properties. So, instead of talking of length-in-meters homomorphisms and mass-in-grams homomorphisms, we can talk of homomorphisms$_1$, where by '$_1$' we indicate that a homomorphism must satisfy a certain list of restrictions. Then, our first claim will be true just in case there is a homomorphism$_1$ that takes meter sticks to 1 and the diameter of Sputnik to .584. Similarly, the second claim will be true if and only if there is a homomorphism$_1$ that takes standard grams to 1 and Sputnik to 83500.

As you might imagine, giving an analysis of the path of Sputnik is slightly more complicated. One way to do this would be to bring in a co-ordinate system. Each spatial point (at a time) would be associated with a triple of numbers, where the three positions would stand for the standard x, y, and z Cartesian co-ordinates in meters. Suppose that $(0, 0, 0)$ was the point of liftoff of the rocket that launched Sputnik into orbit at $t = 0$. To say, then, 'At time t, Sputnik is at position (x, y, z)' is just to say that there is a homomorphism of the appropriate kind from points of space to $\mathbb{R}^3$ that takes Sputnik to (x, y, z) at that time. We have just seen how to handle ordinary length-in-meters claims, and the claims involving the co-ordinate system can be handled by extending these techniques. Again, the idea is that the homomorphism must preserve the relevant properties of the physical situation, here the arrangement of points of space. These restrictions can be enumerated using measurement theory.[25] Once they are in place, we can say that the claim about Sputnik's position will be true if and only if there is an appropriate homomorphism that takes Sputnik to this triple at time t.

4.2 Mappings

Hopefully this discussion has given some sense of the promise of the structuralist approach. Each kind of application, of course, needs a different kind of a mapping, but it does offer a unified account of mixed statements. The central issue for the structuralist proposal, though, is the status of the mappings themselves. We have seen that applications require mappings like isomorphisms of differing cardinality and various kinds of homomorphisms for the different kinds of magnitudes. What are these things supposed to be?

First I will set out my own proposal for how to think about these mappings, and then I will turn to a consideration of the problems that a natural alternative proposal encounters. I claim that these mappings are relations. These relations are like properties. We think of objects having properties, say being red, but we do not think that the property is reducible to the set of the objects that happen to have that property. There is an entity, the property of being red, that we can usefully distinguish from this set. There are many reasons for making this kind of distinction.[26] Perhaps the most

[25] For one way of working out these restrictions see Suppes *et al.* [1989], chapter 14. There are subtle issues connected with space and time generally that I am glossing over here for the purposes of exposition.

[26] See Swoyer [1999] and the introduction to Mellor and Oliver [1997] for a survey of

important is that we want to distinguish properties even if they happen to apply to exactly the same objects. For example, there is no object that is made of gold and bigger than the sun, and there is no object that has read every philosophy paper ever published. However, we do not want to say that these two properties are the same, perhaps because a person can believe that nothing is made of gold and bigger than the sun and have no opinion about whether or not someone has read every philosophy paper. The same point carries over to relations. There is no pair of objects that stands in a relation where both are made of gold and the first is ten billion times bigger than the second, and there is no pair of objects that stands in a relation where both are made of silver and the first is one hundred billion times bigger than the second. For the same reasons as with properties these relations need to be distinguished, and so relations will not be identified with sets containing the pairs that satisfy the relations.

My suggestion, then, is that the mappings involved in applications of mathematics are relations, where these are distinguished from the set of ordered n-tuples that happen to stand in that relation, or what we sometimes call the relation's extension. There is a great deal of debate, though, about exactly what properties and relations are. For my purposes I do not need to take a stand on this issue, beyond saying that relations bear an external relation to their relata. That is, the very same relation will exist regardless of which objects it has as relata. We will see the importance of this shortly. One account of relations that fits in naturally with this requirement is the one contained in Bealer's *Quality and Concept* [1982].[27] I have adopted Bealer's account because it is quite easy to see that he provides enough relations of the kind needed for a structuralist account of applications to work. I remain open to the possibility that a more modest account of relations is available. This is particularly important for adjudicating between the internal-relation solution discussed above, with its attendant modal realism, and the realism about properties that goes with a structuralist approach to applications. Suffice it to say that further research is needed to map out the options available to the structuralist.

Bealer introduces some primitive properties and relations and shows how new intensional entities can be defined in terms of these primitives. In chapter two of Bealer [1982] he sets out a formal language L_ω and two formal derivation systems for two different conceptions of intensional entities such as properties, relations, and propositions. At the heart of Bealer's proposal is an algebraic conception of intensional entities, whereby new intensions can be introduced in terms of other intensions using algebraic operations such as conjunction and negation. For example, using square brackets to

these reasons and the different accounts of properties that result.

[27] See also Bealer [1994] for the contrast between Bealer's type-free approach and typed approaches to properties.

denote an intensional entity, if $[A]_{u_1,\ldots,u_n}$ and $[B]_{v_1,\ldots,v_m}$ are, respectively, n- and m-place relations, we can introduce $[A\&B]_{u_1,\ldots,u_n,v_1,\ldots,v_m}$ as an $(n+m)$-place relation and $[\neg A]_{u_1,\ldots,u_n}$ as the 'negation' of $[A]_{u_1,\ldots,u_n}$. Bealer then develops a model theory for this language in which these terms are assigned appropriate intensional entities as their interpretations. His first conception fits in most naturally with my proposal for mappings being relations. The basic condition on models for this first conception is

$$(\forall x, y \in \mathcal{D}_i)((\forall H \in \mathcal{K})(H(x) = H(y)) \supset x = y), \text{ for all } i \geq -1, \tag{5}$$

where each $\mathcal{D}_i$ is a set of entities of one kind, like two-place relations, and $\mathcal{K}$ is the set of functions H that take an intensional entity to its possible extensions (Bealer [1982], p. 52). For example, if x is a proposition then $H(x)$ is a truth value, and if x is a property then $H(x)$ is a subset of the individuals in the domain. A model that satisfies this condition identifies two intensional entities when their extensions are identical for every H. This captures a 'coarse-grained' criterion of identity for intensional entities that identifies entities if they are necessarily equivalent. This is sufficient for our purposes because the relations involved in applications do not need to be distinguished in any more fine-grained way.[28] Bealer's account of intensional entities fits in well, then, with the account of relations that I want to give: these relations exist independently of whatever entities they happen to relate.

With this theory of intensional entities in place, it is easy to introduce the relations needed for the applications of mathematics that I have considered. For each natural number n, there will be an isomorphism relation that relates objects to the natural numbers. We need a different isomorphism relation for each natural number, and it is convenient to label these with a subscript. Isomorphism$_n$ relates, among other things, the objects in its domain to the natural numbers from 1 to n in a one-one and onto way. With these relations defined, it is possible to give a compact summary of the structuralist analysis of cardinality claims: 'There are n Fs' is true if and only if there is an isomorphism$_n$ relation that relates the Fs to the natural numbers from 1 to n. Similarly, various homomorphism relations can be introduced using the different restrictions that we saw were needed for a homomorphism to preserve a given magnitude. That is, there will be homomorphism$_1$ relations defined as above that will preserve magnitudes like mass and length. Other homomorphisms can be defined for other kinds of magnitudes. We can again give a brief formulation of the structuralist account of magnitude claims: 'Object a is x magnitude-in-units' is true if

[28] Bealer goes on to develop a more fine-grained kind of intensional entity with different criteria of identity, but these aspects of Bealer's system are not relevant to my account of applications.

and only if there is a homomorphism$_M$ relation that relates the standard unit to 1 and a to x, where the homomorphism is the appropriate one for this kind of magnitude.

Now I have not given any argument showing that this approach to applications can handle all applications of mathematics. However, it is important to see to what extent this approach, if it can be generalized further, satisfies both horns of Dummett's dilemma. As the relations are external, what happens in the physical world does not affect the mathematical objects. For this reason, mathematics remains independent of the physical world. At the same time, the isomorphisms and the homomorphisms relate physical objects to mathematics automatically, *i.e.* just in case the physical objects themselves are related in an appropriate way. This gives us the guarantee that mathematics will apply to the physical world, and so we can see in what sense applications are 'direct'.

5. Further Issues

5.1 An Objection

I have been quite candid about the need that the structuralist account has for relations, where these are distinct from what we ordinarily call the relation's extension. Intensional entities are viewed with some suspicion, though, and one might think that it is a serious defect of my account that it brings in this kind of intensional entity. Why not, the objection would go, identify the mappings involved in applications with sets? If such an identification were made, then we would still have all the virtues of the structuralist account of applications but none of the intensional entities that we might worry about. This is the second kind of view that could be attributed to Steiner ([1998], p. 13). He explicitly says that the mappings involved in applications are to be identified with sets. If he also added that mathematical objects such as natural or real numbers were not sets, or at least not sets with physical objects in their transitive closures, then he would have a kind of mapping account without intensional entities.

I have already mentioned the general considerations that often lead philosophers to distinguish intensional entities from their extensions. There are more specific reasons, though, why we should not identify the mappings with sets when these mappings are used to account for applications. These reasons are easy to summarize, although the details get a bit complicated. Briefly, we should not identify relations with sets because to do so would more or less turn the structuralist account into an internal-relation account of the kind discussed earlier. We have already seen that this kind of internal-relation account leads to realism about possible objects. Now, I want to set out why identifying mappings with sets also leads to these unanticipated consequences, even if we do not set up an internal relation between any mathematical objects and the physical world.

To see the problem we first need to review how we might try to identify the mappings needed for applications with sets. The idea is to have a set that has ordered pairs as members. For each pair of objects that the relation relates we will have an ordered pair in the set, with the first object in the first position of the ordered pair and the second object in the second position of the ordered pair. If we call $\mathcal{R}$ the set that has as ordered pairs all and only the pairs that relation R relates, then we will have aRb if and only if $\langle a, b \rangle \in \mathcal{R}$. The key thing to see here is that physical objects must be in the first position of these ordered pairs if the relation is to relate physical objects to mathematical objects. In other words, the sets that are identified with relations will have physical objects in their transitive closure. We have already seen that this is an internal relation. Now, though, we do not have an internal relation between a mathematical object like a natural number and a physical object, but only an internal relation between the sets that we are identifying with relations and physical objects. It may not be immediately apparent that this proposal is committed to realism about possible objects, but I will now set out an example that makes this commitment manifest.

Consider, then, our reasoning about our proposed satellite in 1957. When I say in 1957 'There could have been two American satellites in orbit now if we had had more funding', I am claiming that there could have been American satellites that were related by an isomorphism to the natural numbers 1 and 2. If relations are sets, though, then it looks as though I am saying that there is a set that could have had these satellites in its transitive closure. However, no set actually has these satellites in its transitive closure, and because we have here an internal relation, there is no set that could have had these satellites in its transitive closure. So, it seems that we are forced to give up the structuralist account or introduce possible objects into our set theory. The former option leaves us without an account of applications, while the latter option amounts to realism about possible objects.

As with my original objection to the internal-relation approach, it might seem that I have ignored the obvious solution to these problems. That is to say that the above claim requires only that there could have been an isomorphism between the American satellites and 1 and 2, and this requires only that there could have been a set with the American satellites in its transitive closure. However, if this is right, then we do not need to pick an actual set and say that it would have had different members if the American satellites had existed. For if these satellites had existed it seems clear that there would have been new sets! And, the response concludes, it is these sets that are involved in our modal reasoning.

My objection to this proposal is that it relies on an intensional entity. To see this, note that it is not enough just to say that if there were American satellites then they would be in the transitive closure of some set. I must also add that they will be in the transitive closure of a set that is identified with an isomorphism relation. That is, I must say what kind of set would result if these American satellites were to exist. It is here, though, that I am relying on an intensional entity, which is in this case the property of being a certain kind of set. It is this property which must exist independently of whether or not any sets have the property, and I use it to determine what would be the case if different individuals were part of the set theory. Intensional entities are needed, then, if we are to have a structuralist account that stops short of realism about possible objects. As the motivation for identifying relations with sets was to get rid of intensional entities, and this is not possible, I suggest that we adopt the more perspicuous account that accepts relations as intensional entities from the beginning.

If we identify relations with intensional entities then we easily obtain a clear account of applications involving modal reasoning. For example, in the above case, the isomorphism_2 relation would relate the American satellites if they existed to the natural numbers 1 and 2. As the relations can be held fixed as their extensions vary we can embed them in modal contexts without incident.

5.2 $\in$

A reasonable reply to all is this is to question my starting assumption, namely that a set is internally related to its members. The concept of an internal relation is clearly a modal concept as it involves the impossibility of certain states of affairs. I have claimed that if a set T has c as one of its members, then it is impossible for T not to have c as one of its members. However, to the extent that no modal operators appear in the usual axioms of set theory, one might worry that I have assumed too much. The axiom of extensionality tells us only that if two sets have the same members, then they are identical. This leaves open what is possible or impossible, and in particular it does not force us to conclude that set membership is an internal relation.

The study of the various alternative ways of combining set theory and modal operators is known as 'modal set theory'. My discussion here is based on the work of Charles Parsons.[29] One issue that Parsons discusses is the distinction between an 'existence dependent' conception of set membership and an 'existence independent' conception. Using free logic, with 'E' the predicate for existence, the existence-dependent conception is summarized by three principles:

[29] See part III of Parsons [1983], especially essay 11, and his [1995]. Kit Fine's [1981] is also of interest. I am grateful to one of the anonymous referees for emphasizing the importance of these discussions.

(D1) $x \in y \rightarrow \Box\,(Ey \rightarrow x \in y)$
(D2) $(\neg x \in y \wedge Ey) \rightarrow \Box\,(\neg x \in y)$
(D3) $x \in y \rightarrow (Ex \wedge Ey)$ (Parsons [1983], p. 300).

These principles require that if an object is a member of a set then both the object and the set exist. Furthermore, they stipulate that the membership relations that obtain between objects and sets that exist are necessary. The alternative existence-independent conception stops short of this:

(I1) $x \in y \rightarrow \Box\,(x \in y)$
(I2) $\neg x \in y \rightarrow \Box\,(\neg x \in y)$
(I3) $(Ey \wedge x \in y) \rightarrow Ex$ (Parsons [1983], p. 299).

Parsons notes how set membership is now akin to identity: just as we grant that $x = y \rightarrow \Box\,(x = y)$ even though x and y may not actually exist, so too should we accept (I1) and (I2). Their intended import 'is that a set y in another possible world should have exactly the same elements, whether or not these objects exist, and indeed whether or not y itself exists' (Parsons [1983], p. 299). By contrast, an existence-dependent conception of membership is incompatible with membership relations' obtaining at possible worlds where either the object or the set fail to exist.

My discussion has assumed an existence-dependent conception of set membership. This is reflected in the steps in my arguments from the non-existence of an object, *e.g.* an American satellite, to the conclusion that the object is not a member of any set. This move could be underwritten by the contrapositive of (D3): $\neg Ex \vee \neg Ey \rightarrow \neg(x \in y)$. No analogous move is licensed by the existence-independent conception. x's failure to exist at a possible world tells us only that either the set y also fails to exist there or that it is not the case that $x \in y$. Thus one could escape the arguments I have presented by adopting an existence-independent conception of set. Then the two American satellites that could have existed, but do not, would still be members of the transitive closure of the natural number two. In effect, we make possible objects just as important as actual objects in fixing the identity of a set. Such a view is consistent with extensionality because we still individuate sets in terms of their members. And, it stops short of modal realism because the possible objects can be members of sets without actually existing.[30]

5.3 Evolving Hybrid Accounts

A second way out is to question the significance of a wholly internal-relation approach to applications. The problems that such an account runs into

[30] More subtle complications arise once we recall that natural numbers are proper classes according to the account I ascribe to Frege and Steiner. Parsons's investigations of modal set theory have been partly motivated by his desire to clarify the distinction between set and class, and in particular the role of the iterative conception of set in motivating the axioms of ZF. It may very well be that no conception of membership for proper classes is consistent with (D1)–(D3), but I must reserve this debate for future work.

when handling modal claims suggest that nobody, including Frege and Steiner, could have intended to present it. Rather, in terms of my three-part classification scheme of no-relation, internal-relation and external-relation, any reasonable account would be a hybrid account. So, my argument against the set-theoretic account deals only with a position that nobody would hold. More seriously, if my classification scheme ignores plausible hybrid positions, then it loses whatever value it might have had as it no longer clarifies the positions of the genuine players in the debate about applications. Perhaps we should go back to thinking of accounts of applications in terms of the more traditional platonist/nominalist positions.

To see the promise of a hybrid account that is in the spirit of the approaches of Frege and Steiner, imagine that the natural numbers are identified with the proper classes of all sets that have just that number of objects as members, where only actual objects are considered as individuals. For claims about the actual world, we have the straightforward internal-relation account in terms of set membership. However, given that possible objects are not members of these sets, we cannot explain applications of the natural numbers in counting possible objects in the same way. Still, there seems to be no problem with adopting an external-relation approach to handling these applications. If the structuralist approach will work in all cases, it will of course also work for this more limited range of applications. That is, we interpret the claim 'There could have been two American satellites' in terms of the existence of an isomorphism$_2$ relation in some possible world that relates the two American satellites to 1 and 2. Instead of the purely external-relation approach canvassed above, though, now 1 and 2 have some definite internal structure that is used in handling some applications. But because these internal relations are not used to handle applications in modal contexts, there is no commitment to modal realism.[31]

My first point is that the existence of hybrids does not undermine the value of my classification scheme. On the contrary, it gives us a clear sense of what sorts of internal-relation approaches are viable, when they become committed to modal realism, *etc*. Other approaches to applications also fall into the internal-relation category, and will be vulnerable to this modal objection. More interestingly, there are other types of objections to internal-relation accounts as a group that push one either towards a hybrid account or a purely external approach. As it is beyond the scope of this paper to present these positions or objections in any detail, a brief summary will have to suffice. Briefly, if the subject matter of mathematics is just empirical generalities, as it is for Mill and to a certain extent Kitcher ([1984]), then there is an internal relation between the subject matter of mathematics and the situations to which mathematics is applied. I claim

[31] I would like to thank a second referee for the suggestion that I explicitly consider this sort of hybrid approach in this paper.

that both a Fregean approach and such empiricist approaches have troubles with modal contexts. Furthermore, I believe that they both have serious difficulties explaining applications of mathematics within mathematics.

The goal of this paper, then, is not just to complain about some problem with a view that I find in Frege. It is rather to reconsider the options for thinking about applications and to try to constrain these options using arguments that rule out approaches that seem viable. At the end of the day, hybrids will no doubt have to be considered along with purely external and internal accounts, as will the alternative conceptions of set membership reviewed above. What is surprising to me is that no such systematic work has been undertaken to date.

6. Conclusion

I have claimed that a wholly internal-relation approach requires modal realism if it is to be of any use interpreting descriptions of counterfactual situations. Opinions, of course, differ on the need for such an interpretation, but I am assuming in this essay that we cannot follow Quine's dismissal of modal talk. Still, the force of my objection to internal approaches seems considerably diminished by the rich ontology introduced in the course of spelling out the details of the structuralist approach. We saw first that a realistic attitude towards properties and properties of properties is needed in specifying our mappings. And, later I also introduced new relations to explain the nature of the mappings themselves. Even if we can avoid modal realism with this assortment of properties and relations, it seems like something of a Pyrrhic victory.

In order to assess the strengths of the structuralist position it is first important that we see that it is compatible with non-realist interpretations of modal talk. The main alternatives to modal realism invoke *abstracta* such as states of affairs.[32] These states of affairs are all actual, but only some of them are instantiated in the actual world. Modal talk involves claims about what would be the case if various ranges of states of affairs were instantiated. In the above example, there is the state of affairs of there being two American satellites. In the actual world it is not instantiated. But, were it to be instantiated, there would be a mapping from the two satellites to the numbers 1 and 2. The very same mapping exists regardless of which states of affairs obtain, and so we can hold this mapping fixed as different instantiations of states of affairs obtain.

As we have also taken a realistic attitude towards length properties, the structuralist account can also interpret modal talk of the lengths of possible objects in terms of states of affairs. For example, if there were to be two American satellites, then let us assume that one would be longer

[32] See Loux [1979], Chihara [1998], and Loux [1998], ch. 5 for further discussion of these issues.

than the other. That is, if one such state of affairs was instantiated we would have two new objects with differing length properties. In virtue of these length properties, the objects would be mapped by a length-in-meters homomorphism to differing real numbers. And, of course, there will always be a greater real number for any pair of different real numbers. Here both the realism about length properties and mappings is brought into play in saying what would be the case in some counterfactual situation.

I do not wish to suggest that the flexibility of the structuralist account automatically shows that it is superior to any internal-relation account. There is a clear tradeoff, though, between accepting properties and being able to avoid modal realism. If one were already committed to modal realism, then these problems with the internal-relation approach would be easily resolved. Here I have argued only that the structuralist account is a promising approach to develop. I have, of course, not shown that it is the only account worth pursuing or even that it is the only viable account that falls in the 'external-relation' category. For example, an account that identified mathematical entities with intensional entities might have some advantages of economy over the structural account.[33] As I noted at the end of section 5, I hope that the framework provided in this essay will spur further investigation into different approaches to applications of mathematics and a more nuanced recognition of the costs of these approaches.

[33] See, *e.g.*, Bigelow [1988].

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ABSTRACT. This paper sets out a new framework for discussing a long-standing problem in the philosophy of mathematics, namely the connection between the physical world and a mathematical domain when the mathematics is applied in science. I argue that considering counterfactual situations raises some interesting challenges for some approaches to applications, and consider an approach that avoids these challenges.